Modern Algebra
Spring 2019

Assignment 11.1
Due April 17

Exercise 1. Let A and B be abelian groups, and let $f : A \to B$ be a homomorphism. Suppose there is a homomorphism $g : B \to A$ so that $f \circ g = 1_B$. Prove that $A = \ker f \oplus \operatorname{im} g$. [*Suggestion:* Given $a \in A$, write $a = (a - g(f(a))) + g(f(a))$.]

Exercise 2. Let A be an additive finite cyclic group of order n. For each $m \in \mathbb{Z}$ with $\gcd(m, n) = 1$ define $\sigma_m : A \to A$ by $\sigma_m(a) = ma$.

- **a.** Prove that $\sigma_m \in \operatorname{Aut}(A)$. [*Suggestion:* Use the fact that $\gcd(m, n) = 1$ to show that $\ker \sigma_m = \{0\}$. Since A is finite, this implies σ_m is onto (why?).]
- **b.** Prove that $\sigma_m = \sigma_\ell$ if and only if $m \equiv \ell \pmod{n}$. [*Suggestion:* For one implication, apply the two automorphisms to a generator of A.]
- **c.** Use parts **a** and **b** to construct a monomorphism $(\mathbb{Z}/n\mathbb{Z})^\times \to \operatorname{Aut}(A)$. [*Remark:* This is *not* just a consequence of the first isomorphism theorem, since $(\mathbb{Z}/n\mathbb{Z})^\times$ is not a quotient of $\mathbb{Z}$, but a subset of a quotient.]
- **d.** Show that the map of part **d** is an isomorphism. [*Suggestion:* If A is generated by x, and $\sigma \in \operatorname{Aut}(A)$, then $\sigma(x) = mx$ for some integer m. Use the fact that $|\sigma(x)| = |x|$ to show that $\gcd(m, n) = 1$.]

Remark. A somewhat more natural, but more advanced, approach to Exercise 2 that *does* take advantage of the first isomorphism theorem replaces $\operatorname{Aut}(A)$ with $\operatorname{End}(A)$, the *ring* of arbitrary homomorphisms $A \to A$. The map $m \mapsto \sigma_m$ yields a surjective ring homomorphism $\mathbb{Z} \to \operatorname{End}(A)$, and the ring-theoretic first isomorphism theorem then implies $\mathbb{Z}/n\mathbb{Z} \cong \operatorname{End}(A)$. By restricting this to the *unit groups* on both sides, we get the isomorphism of **2d** above "for free."