Modern Algebra
Spring 2019

Assignment 11.2
Due April 17

Exercise 1. Let C be an (additive) cyclic group of order n, generated by a, and suppose $n = de$ for some $d, e \in \mathbb{N}$. Prove that

$$C_d = \langle ea \rangle .$$

Exercise 2. Let p be a prime and let G be a group. We say that G is a *p-group* if the order of any element of G is a power of p.[1] Prove the following statements about a p-group G.

a. Let $x, y \in G$. If $m = \max\{|x|, |y|\}$, then $x^m = y^m = e$.

b. If $m = \max\{|x| : x \in G\}$ is finite, then m is an exponent for G.

Exercise 3. Given a prime p, an *elementary abelian p-group* is an abelian group in which every nonidentity element has order p.[2]

a. Use the fundamental theorem of finite abelian p-groups (as we called it) to give a classification of all finite elementary abelian p-groups.

b. Can you think of a countably infinite elementary abelian p-group? How about uncountable?

[1] If G is finite and abelian, we have proven that this definition is equivalent to $|G|$ being a power of p. If we assume only that G is finite, the same conclusion holds, but requires a different proof.

[2] Under modular arithmetic (addition *and* multiplication), the factor group $\mathbb{Z}/p\mathbb{Z}$ is a *field*, and the elementary abelian p-groups are precisely the $\mathbb{Z}/p\mathbb{Z}$-vector spaces.