Modern Algebra
Spring 2019

Assignment 12.2
Due May 1

Exercise 1. Let G be a group, S a set, and

$$\begin{aligned} G &\to \operatorname{Perm}(S), \\ x &\longmapsto \alpha_x, \end{aligned}$$

a homomorphism. For $x \in G$ and $s \in S$, define

$$x \cdot s = \alpha_x(s).$$

Show that this is an action of G on S.

Exercise 2. Let $G = \mathrm{SL}_2(\mathbb{R}) = \{A \in \mathrm{GL}_2(\mathbb{R}) \mid \det A = 1\}$ and $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, and $z \in \mathcal{H}$, prove that

$$\gamma z = \frac{az + b}{cz + d}$$

is a well-defined action of G on $\mathcal{H}$. [*Suggestion:* The "hard" part is proving associativity of the action. This can certainly be done directly, but it can also be facilitated by working on $\mathbb{P}^1(\mathbb{C})$, the (projective) Riemann sphere, instead of $\mathcal{H}$.]

Exercise 3. Let G be a group acting on a set S. Any subgroup of G acts on S by restriction, by what about factors of G? Let $N \lhd G$. The natural way to let G/N act on S would be via representatives. That is, we would like to define $(xN) \cdot s = xs$ for $xN \in G/N$ and $s \in S$. But in order for this to be well-defined, we must have $xs = ys$ when $xN = yN$ and $s \in S$, which may not happen.

Prove that if we replace S by the subset $S^N = \{s \in S \mid ns = s \text{ for all } n \in N\}$ of N-*invariants* of S, then the rule above yields a well-defined action of G/N on S^N.[1] Moreover, show that if G/N acts "naturally" on any other $T \subset S$, then $T \subset S^N$.

[1] This result is used in algebraic number theory, when trying to define the Euler factor at a ramified prime for the Artin L-function of a finite dimensional representation of the group of a Galois extension of number fields.