Modern Algebra
Spring 2019

Assignment 3.1
Due February 6

Exercise 1. Given a prime p, let

$$\mathbb{Z}_{(p)} = \left\{ \frac{r}{s} \;\middle|\; r \in \mathbb{Z}, s \in \mathbb{N}, p \nmid s \right\},$$
$$\mathbb{Z}[p^{-\infty}] = \left\{ \frac{r}{p^t} \;\middle|\; r \in \mathbb{Z}, t \in \mathbb{N}_0 \right\}.$$

Prove that $\mathbb{Z}_{(p)}$ and $\mathbb{Z}[p^{-\infty}]$ are (additive) subgroups of $\mathbb{Q}$.

Exercise 2. For $n \in \mathbb{N}$, let $\boldsymbol{\mu}_n = \{z \in \mathbb{C} \mid z^n = 1\}$. Prove that $\boldsymbol{\mu}_n$ is a (multiplicative) subgroup of $\mathbb{C}^\times$.

Exercise 3. Given a complex number $z = x + iy$ ($x, y \in \mathbb{R}$), define its *complex conjugate* to be $\overline{z} = x - iy$. Prove the following properties of conjugation.

- **a.** For all $z, w \in \mathbb{C}$, $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$.
- **b.** For any $z \in \mathbb{C}$, $z\overline{z} = |z|^2$.
- **c.** For any $z \in \mathbb{C}$, $z = \overline{z}$ if and only if $z \in \mathbb{R}$.

Use parts **a** and **b** to show that $|zw| = |z||w|$ for all $z, w \in \mathbb{C}$.

Exercise 4. Let G be a group and suppose that $\mathcal{F}$ is a family of subgroups of G. Prove that

$$J = \bigcap_{H \in \mathcal{F}} H$$

is a subgroup of G.