Modern Algebra
Spring 2019

Assignment 4.3
Due February 13

Exercise 1. Let $f : G \to H$ be a homomorphism of groups.

a. If $J < G$, prove that $f(J) < H$.

b. If $K < H$, prove that $f^{-1}(K) < G$.[1]

Exercise 2. Let G be a group and let $\mathrm{Aut}(G)$ denote the set of automorphisms of G.

a. Prove that $\mathrm{Aut}(G) < \mathrm{Perm}(G)$.

b. Given $x \in G$, define $c_x : G \to G$ by $c_x(y) = xyx^{-1}$. Prove that $c_x \in \mathrm{Aut}(G)$. The map c_x is called *conjugation by* x.

c. Prove that the map $x \mapsto c_x$ from G to $\mathrm{Aut}(G)$ is a homomorphism. Its image is subgroup of *inner automorphisms* of G.

Exercise 3. Let $\phi : \mathbb{Q} \to \mathbb{Q}$ be an endomorphism of $\mathbb{Q}$. Prove that there exists an $a \in \mathbb{Q}$ so that $\phi(r) = ar$ for all $r \in \mathbb{Q}$. [*Suggestion:* Consider $\phi(1)$.]

[1] Recall that given any function $f : X \to Y$ and any subset $S \subset Y$, the *inverse image* of S under f is

$$f^{-1}(S) = \{x \in X \mid f(x) \in S\}.$$

The inverse image makes sense whether or not f is invertible. However, when f does happen to be invertible, $f^{-1}(S)$ is indeed the same as the forward image of S under f^{-1}, as the notation suggests.