**Exercise 1.** Let $A$ be an (additive) abelian group, $B_1, B_2, \ldots, B_n < A$. Define

$$f : B_1 \times B_2 \times \cdots \times B_n \to A$$

by $f(b_1, b_2, \ldots, b_n) = b_1 + b_2 + \cdots + b_n$.

**a.** Prove that $f$ is a homomorphism. Use this to conclude that

$$B_1 + B_2 + \cdots + B_n = \{b_1 + b_2 + \cdots + b_n \mid b_i \in B_i\}$$

is a subgroup of $A$.

**b.** Prove that $\ker f = \{0\}$ if and only if $(B_1 + B_2 + \cdots + \widehat{B_i} + \cdots + B_n) \cap B_i = \{0\}$ for all $1 \le i \le n$. Here the caret indicates the omission of the summand $B_i$.

**c.** Prove that $\ker f = \{0\}$ if and only if $B_i \cap (B_{i+1} + \cdots + B_n) = \{0\}$ for all $1 \le i \le n - 1$.

When $\ker f = \{0\}$, $f$ provides an isomorphism between the *external* product $B_1 \times B_2 \times \cdots \times B_n$ and the *internal* sum $B_1 + B_2 + \cdots + B_n$. In this case we call the latter the *(internal) direct sum* and denote it by $B_1 \oplus B_2 \oplus \cdots \oplus B_n$.

**Exercise 2.** With $A$ and $B_i$ as above, prove that

$$B_1 \oplus B_2 \oplus B_3 = B_1 \oplus (B_2 \oplus B_3) = (B_1 \oplus B_2) \oplus B_3,$$

i.e. the direct sum is associative. Yes, there really is something to prove here.

**Exercise 3.** Let $f : G \to H$ be a group homomorphism.

**a.** Prove that for any $a \in G$, the order of $f(a)$ divides the order of $a$.

**b.** Prove that if $f$ is a monomorphism, then for any $a \in G$, $a$ and $f(a)$ have the same order.

**c.** Use part **b** (and an earlier result) to provide an alternate proof of the fact that for any $a, b \in G$, $a$ and $bab^{-1}$ have the same order.