Exercise 1. Prove that $\mathbb{Q}$ has no proper subgroups of finite index.

Exercise 2. Prove that $\mathbb{Q}/\mathbb{Z}$ has elements of every finite order.

Exercise 3. Let G be a group, $S \subset G$ and $H = \langle S \rangle$. Fix $x \in G$ and set $K = \langle xSx^{-1} \rangle$.

- **a.** Prove that $K = xHx^{-1}$. [*Suggestion:* Prove that the subgroups of G containing xSx^{-1} are precisely the x-conjugates of the subgroups containing S.]
- **b.** Conclude that if $xSx^{-1} \subset S$ for all $x \in G$, then $H \lhd G$.

Exercise 4. Given a group G, its *commutator subgroup* is defined to be

$$G' = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle.$$

- **a.** Prove that $G' \lhd G$. [*Suggestion:* Use the preceding exercise.]
- **b.** If $H \lhd G$, prove that G/H is abelian if and only if $G' < H$. Thus G/G' is the largest abelian factor group of G.
- **c.** If $G' < H < G$, prove that $H \lhd G$.