Modern Algebra
Spring 2019

Assignment 8.1
Due March 20

Exercise 1. Let G_1 and G_2 be groups, with $H_1 \lhd G_1$ and $H_2 \lhd G_2$. Use the First Isomorphism Theorem to prove that $H_1 \times H_2 \lhd G_1 \times G_2$ and

$$(G_1 \times G_2)/(H_1 \times H_2) \cong (G_1/H_1) \times (G_2/H_2).$$

[*Suggestion:* Use the canonical epimorphisms $G_i \to G_i/H_i$ to construct a homomorphism $G_1 \times G_2 \to (G_1/H_1) \times (G_2/H_2)$, and compute its kernel.]

Exercise 2. Let $m, n \in \mathbb{N}$ be relatively prime. Use the First Isomorphism Theorem to prove the Chinese Remainder Theorem:

$$\mathbb{Z}/mn\mathbb{Z} \cong (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}).$$

[*Suggestion:* Use the canonical epimorphisms from $\mathbb{Z}$ onto $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ to construct a homomorphism $\mathbb{Z} \to (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$, and compute its kernel. Use the Pigeonhole Principle to prove that the induced monomorphism is surjective.]

Exercise 3. Let G be a group, $H < G$ and $N \lhd G$. Prove the Second Isomorphism Theorem:

$$H/(H \cap N) \cong HN/N.$$

[*Suggestion:* Compose the inclusion $H \hookrightarrow HN$ with the canonical surjection $HN \to HN/N$, and compute the kernel. You'll need a separate argument to show that the induced monomorphism is surjective.]