Exercise 1. Let G be a group and $H < G$. Recall that for each $x \in G$, the left-translation map $T_x : G/H \to G/H$[1] given by $yH \mapsto xyH$ is an element of $\mathrm{Perm}(G/H)$, and that the map $x \mapsto T_x$ is a homomorphism $G \to \mathrm{Perm}(G/H)$. Let K denote the kernel of this homomorphism.

Assume that G is finite and $[G : H] = p$ is the smallest prime dividing $|G|$.

a. Use the first the First Isomorphism Theorem to show that $[G : K]$ divides p.

b. Use the fact that $K < H$ to show that $[G : K]$ is divisible by p.

c. Conclude that $[G : K] = p$ and use this to show $H \lhd G$.

Exercise 2. Let G be a nonabelian group of order 8.

a. Explain why G must have an element of order 4.

b. Let $x \in G$ have order 4 and $H = \langle x \rangle$. Show that $[G : H] = 2$ and conclude that $H \lhd G$.

c. Let $y \in G \setminus H$. Show that $G = \langle x, y \rangle$ and $y^2 \in H$. Conclude that $y^2 = e$ and $|y| = 2$, or $y^2 = x^2$ and $|y| = 4$. [*Suggestion:* Show that $y^2 = x$ and $y^2 = x^3$ both imply $x^2 = e$.]

d. Prove that $yxy^{-1} = x$ or $yxy^{-1} = x^3$. Show that the former is impossible since it implies G is abelian. [*Suggestion:* H is normal in G and conjugation preserves orders of elements.]

Exercise 3. Let everything be as in the preceding exercise.

a. If $y^2 = e$, prove that G is isomorphic to the dihedral group

$$D_4 = \langle r, f \,:\, |r| = 4, |f| = 2, frf = r^{-1} \rangle.$$

b. If $y^2 = x^2$, prove that G isomorphic to the quaternion group

$$Q = \langle i, j \,:\, |i| = |j| = |ij| = 4, i^2 = j^2 = (ij)^2 \rangle.$$

[1] Here G/H simply indicates the set of left cosets of H in G. We are *not* assuming $H \lhd G$.