# Countable Sets

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Intro to Abstract Mathematics

# Countability

**Definition**

Let $X$ be a set. We say that $X$ is *countable* if it is possible to list the elements of $X$:

$$X = \{x_1, x_2, x_3, x_4, \ldots\}.$$

If the list eventually ends we say $X$ is *finite*. If the list goes on indefinitely we say $X$ is *countably infinite*.

A finite (countable) set has the form

$$X = \{x_1, x_2, x_3, \ldots, x_n\},$$

where $n \in \mathbb{N}^+$ is the number of elements of $X$. In this case we write

$$|X| = n.$$

The empty set is vacuously countable:

$$\varnothing = \{\,\},$$

and we write $|\varnothing| = 0$.
If $X$ is countably infinite we write

$$|X| = \aleph_0$$

(pronounced *aleph naught*).

**Example 1**

The sets $\{\pm 1\}$, $\{\pi, \sqrt{3}, e, \sqrt{5}\}$, $\{1, 2, 3, 4, \ldots, 1000\}$ are all finite.

**Example 2**

The sets $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$, $E = \{2, 4, 6, 8, 10, \ldots\}$ and $P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$ are countably infinite.

Countability can be rephrased in terms of certain bijections. For $n \in \mathbb{N}^+$ let

$$I(n) = \{1, 2, 3, 4, \ldots, n\}.$$

**Theorem 1**

*Let $X$ be a nonempty set.*

1. *$X$ is finite iff there is an $n \in \mathbb{N}^+$ and a bijection $f : I(n) \to X$.*
2. *$X$ is countably infinite iff there is a bijection $g : \mathbb{N}^+ \to X$.*

*Proof.* If $X$ is finite with $|X| = n$, then

$$X = \{x_1, x_2, x_3, \ldots, x_n\}.$$

The rule $f(k) = x_k$ is a bijection $f : I(n) \to X$.

If $X$ is countably infinite, then

$$X = \{x_1, x_2, x_3, \ldots\},$$

and the rule $g(k) = x_k$ is a bijection $g : \mathbb{N}^+ \to X$. $\square$

**Example 3**

The set $\mathcal{P}(\mathbb{N}^+)$ is *not* countable.

*Proof.* $\mathcal{P}(\mathbb{N}^+)$ is not finite since $\{n\} \in \mathcal{P}(\mathbb{N}^+)$ for all $n \in \mathbb{N}^+$.

And according to Cantor's theorem, there is no bijection $f : \mathbb{N}^+ \to \mathcal{P}(\mathbb{N}^+)$, either. $\square$

# More Examples

## Example 4

The set $\mathbb{Z}$ is countable since

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \ldots\}.$$

## Example 5

The set $\mathbb{N} \times \mathbb{N}$ is countable since we can display its elements as a grid and list them diagonally:

$$\begin{array}{cccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & \cdots \\
(1,0) & (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\
(2,0) & (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
(3,0) & (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
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x_3 & (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\
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x_3 & x_5 & (1,2) & (1,3) & (1,4) & \cdots \\
(2,0) & (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
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x_1 & x_2 & x_4 & (0,3) & (0,4) & \cdots \\
x_3 & x_5 & (1,2) & (1,3) & (1,4) & \cdots \\
x_6 & (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
(3,0) & (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
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$$\begin{array}{cccccc}
x_1 & x_2 & x_4 & x_7 & (0,4) & \cdots \\
x_3 & x_5 & (1,2) & (1,3) & (1,4) & \cdots \\
x_6 & (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
(3,0) & (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
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$$\begin{array}{cccccc}
x_1 & x_2 & x_4 & x_7 & (0,4) & \cdots \\
x_3 & x_5 & x_8 & (1,3) & (1,4) & \cdots \\
x_6 & (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
(3,0) & (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
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x_1 & x_2 & x_4 & x_7 & (0,4) & \cdots \\
x_3 & x_5 & x_8 & (1,3) & (1,4) & \cdots \\
x_6 & x_9 & (2,2) & (2,3) & (2,4) & \cdots \\
(3,0) & (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
(4,0) & (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\
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x_3 & x_5 & x_8 & (1,3) & (1,4) & \cdots \\
x_6 & x_9 & (2,2) & (2,3) & (2,4) & \cdots \\
x_{10} & (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
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$$\begin{array}{cccccc}
x_1 & x_2 & x_4 & x_7 & x_{11} & \cdots \\
x_3 & x_5 & x_8 & (1,3) & (1,4) & \cdots \\
x_6 & x_9 & (2,2) & (2,3) & (2,4) & \cdots \\
x_{10} & (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
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$$\begin{array}{cccccc}
x_1 & x_2 & x_4 & x_7 & x_{11} & \cdots \\
x_3 & x_5 & x_8 & x_{12} & (1,4) & \cdots \\
x_6 & x_9 & (2,2) & (2,3) & (2,4) & \cdots \\
x_{10} & (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
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x_{10} & (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
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x_3 & x_5 & x_8 & x_{12} & (1,4) & \cdots \\
x_6 & x_9 & x_{13} & (2,3) & (2,4) & \cdots \\
x_{10} & x_{14} & (3,2) & (3,3) & (3,4) & \cdots \\
(4,0) & (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
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$$\begin{array}{cccccc}
x_1 & x_2 & x_4 & x_7 & x_{11} & \cdots \\
x_3 & x_5 & x_8 & x_{12} & (1,4) & \cdots \\
x_6 & x_9 & x_{13} & (2,3) & (2,4) & \cdots \\
x_{10} & x_{14} & (3,2) & (3,3) & (3,4) & \cdots \\
x_{15} & (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

# Uncountability of $\mathbb{R}$

A somewhat amazing fact is that there are more real numbers than there are natural numbers, in the following precise technical sense.

**Theorem 2**

*The set $\mathbb{R}$ is uncountable.*

*Proof.* It suffices to prove the interval $(0,1)$ is uncountable.

We use *Cantor's diagonal argument*.

Every $x \in (0,1)$ has a decimal representation

$$x = 0.d_1 d_2 d_3 d_4 \ldots, \quad \text{with } d_k \in \{0,1,2,\ldots,9\},$$

which is unique if we *do not allow terminating strings of 9s*.

We assume $(0,1)$ is countable and derive a contradiction.

So write $(0,1) = \{x_1, x_2, x_3, \ldots\}$ and list the decimal representations:

$$\begin{aligned}
x_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\
x_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\
x_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\
x_4 &= 0.d_{41}d_{42}d_{43}d_{44}\ldots \\
\vdots & \qquad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{aligned}$$

For $d \in \{0, 1, 2, \ldots, 9\}$ define

$$d' = \begin{cases} 1 & \text{if } d = 2, \\ 2 & \text{otherwise.} \end{cases}$$

Notice that $d' \in \{0, 1, 2, \ldots, 9\}$ and $d' \neq d$.

Now define $x$ by using $d'$ to alter the diagonal entries of the table of digits:

$$x = 0.d'_{11}d'_{22}d'_{33}d'_{44}\ldots \in (0, 1).$$

Notice that $x \neq x_k$ for any $k$, since $d'_{kk} \neq d_{kk}$.

So $x \notin \{x_1, x_2, x_3, \ldots\} = (0, 1)$, a contradiction. $\square$