Equivalence Relations

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Intro to Abstract Mathematics

Three Important Types of Relations

Definition

Let A be a set and let R be a relation on A. We say that R is:

1. *Reflexive* if aRa for all $a \in A$;
2. *Symmetric* if aRb implies bRa, for all $a, b \in A$;
3. *Transitive* if aRb and bRc imply aRc, for all $a, b, c \in A$.

Remark. It is not difficult to show that:

Theorem 1

Let R be a relation on a set A. Then:

1. *R is reflexive iff $i_A \subset R$;*
2. *R is symmetric iff $R = R^{-1}$;*
3. *R is transitive iff $R \circ R \subset R$.*

(Recall that $i_A = \{(a, a) \mid a \in A\}$ is the *identity relation*.)

Examples

Example 1

$L_1 = \{(m, n) \in \mathbb{Z}^2 \mid m < n\}$ on $\mathbb{Z}$

1. L_1 is *not* reflexive since $(0, 0) \notin L_1$ (for instance).
2. It is *not* symmetric since $(1, 2) \in L_1$, but $(2, 1) \notin L_1$.
3. But it *is transitive*, since $m < n$ and $n < \ell$ together imply $m < \ell$.

Example 2

$D = \{(x, y) \in \mathbb{R}^2 \mid |x - y| < 1\}$ on $\mathbb{R}$

1. D *is reflexive* since $|x - x| < 1$ for all $x \in \mathbb{R}$.
2. It *is symmetric* since $|x - y| = |y - x|$ for all $x, y \in \mathbb{R}$.
3. But it is *not* transitive, since $(0, 3/4), (3/4, 3/2) \in D$ but $(0, 3/2) \notin D$.

Example 3

Let X be a nonempty set and $V = \{(A, B) \in \mathcal{P}(X)^2 \mid A \cap B = \varnothing\}$, a relation on $\mathcal{P}(X)$.

1. V *not* reflexive since $X \cap X \neq \varnothing$.
2. It *is symmetric* since $A \cap B = B \cap A$ for all $A, B \in \mathcal{P}(X)$.
3. But it is *not* transitive, since $(X, \varnothing), (\varnothing, X) \in V$ but $(X, X) \notin V$.

Example 4

Let $m \in \mathbb{N}$ and $C_m = \{(a, b) \in \mathbb{Z}^2 \,:\, m|a - b\}$, a relation on $\mathbb{Z}$.

1. C_m *is reflexive* since $m|a - a$ for all $a \in \mathbb{Z}$.
2. It *is symmetric* since $m|k$ if and only if $m|(-k)$ for any $k \in \mathbb{Z}$.
3. And it *is transitive* since if $m|a - b$ and $m|b - c$, then $m|(a - b) + (b - c) = a - c$, for any $a, b, c \in \mathbb{Z}$.

Equivalence Relations

Definition

A relation R on a set A is called an *equivalence relation* if it is reflexive, symmetric and transitive.

Remarks.

1. An equivalence relation generalizes the notion of strict equality/identity between objects.
2. It is common to denote equivalence relations using more suggestive notation, such as $\cong$, $\equiv$, $\approx$, $\sim$, instead of single letters like R.
3. Objects related by an equivalence relation can be thought of as "the same," in a sense to be made more precise later.

Examples

Example 5

The relation $C_m = \{(a, b) \in \mathbb{Z}^2 : m|a - b\}$ on $\mathbb{Z}$ is an equivalence relation, for every $m \in \mathbb{N}$.

This follows from the final example above.

C_m is called *congruence modulo m*.

Instead of aC_mb one writes $a \equiv b \pmod{m}$.

Let $n \in \mathbb{Z}$ and use the division algorithm to write $n = qm + r$, $0 \leq r < m$.

Then $m|n - r$ so that $n \equiv r \pmod{m}$.

Because remainders are unique, this proves:

Theorem 2

Let $m \in \mathbb{N}$. For each $n \in \mathbb{Z}$ there is a unique $r \in \{0, 1, 2, \ldots, m-1\}$ so that $n \equiv r \pmod{m}$.

Remark. In fact, Theorem 2 is *equivalent* to the division algorithm.

Example 6

Let $C_Z = \{(x, y) \in \mathbb{R}^2 \,|\, x - y \in \mathbb{Z}\}$. Then C_Z is an equivalence relation on $\mathbb{R}$.

The proof is left as an exercise.

Instead of xC_Zy, one writes $x \equiv y \pmod{\mathbb{Z}}$.

Example 7

Let $S = \mathbb{Z} \times \mathbb{N}^+$ and define

$$Q = \{((a, b), (c, d)) \in S^2 \mid ad - bc = 0\}.$$

Show that Q is an equivalence relation on S.

We must show Q is reflexive, symmetric and transitive.

1. *Reflexive:* Let $(a, b) \in \mathbb{Z} \times \mathbb{N}^+$. Then $(a, b)Q(a, b)$ since $ab - ba = 0$.
2. *Symmetric:* Let $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{N}^+$.

   Suppose that $(a, b)Q(c, d)$. Then $ad - bc = 0$.

   Hence

   $$0 = -(ad - bc) = bc - ad = cb - da,$$

   so that $(c, d)Q(a, b)$.

3. *Transitive* Let $(a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{N}^+$.

Suppose $(a, b)Q(c, d)$ and $(c, d)Q(e, f)$. Then $ad - bc = 0$ and $cf - de = 0$.

Consequently,

$$0 = f(ad - bc) + b(cf - de) = fad - bde = (af - be)d.$$

Since $d \neq 0$, we have $af - be = 0$ and thus $(a, b)Q(e, f)$. □

Remark. Notice we only used the fact that second coordinates are positive (as opposed to nonnegative) in the final step.