Functions

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Intro to Abstract Mathematics

Introduction

A *function* is a specific type of relation between two sets.

Definition

Let X, Y be sets. A *function from X to Y* is a relation $f \subset X \times Y$ so that for each $x \in X$ there is a unique $y \in Y$ with $(x, y) \in f$.

Remarks.

1. Uniqueness means that if $(x, y) \in f$ and $(x, y') \in f$, then $y = y'$ (this is the *vertical line test*).
2. If f is a function from X to Y we write $f : X \to Y$, and $f(x) = y$ whenever $(x, y) \in f$. Equivalently, $(x, f(x)) \in f$.
3. Informally, a function is a "rule" that assigns one element of Y to each element of X.

Examples

1. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$. Then

$$f = \{(1, b), (2, a), (3, e), (4, a)\},$$
$$g = \{(1, a), (2, d), (3, b), (4, c)\},$$

are functions $X \to Y$. We write

$$f(1) = b, \quad f(2) = a, \quad f(3) = e, \quad f(4) = a,$$
$$g(1) = a, \quad g(2) = d, \quad g(3) = b, \quad g(4) = c.$$

2. However, with X, Y as above,

$$h = \{(1, a), (2, b), (3, b), (4, c), (1, d)\},$$
$$k = \{(1, c), (2, a), (4, e)\}$$

are *not* functions $X \to Y$ (why?).

Examples

3. Let $X = [0,3] \subset \mathbb{R}$ and $Y = \mathbb{R}$. Then

$$f = \{(x, (x-1)^2) \in X \times Y\}$$

is a function $X \to Y$. We write $f(x) = (x-1)^2$.

4. If A and B are nonempty sets, then

$$\pi_1 = \{((a,b),a) \mid (a,b) \in A \times B\},$$
$$\pi_2 = \{((a,b),b) \mid (a,b) \in A \times B\},$$

are functions, $A \times B \to A$ and $A \times B \to B$, respectively. We have

$$\pi_1(a,b) = a \quad \text{and} \quad \pi_2(a,b) = b.$$

π_1, π_2 are called the *projections* onto the first and second coordinates, respectively.

Domain, Codomain and Range

Definition

Let $f : X \to Y$ be a function. The *domain* of f is

$$\mathrm{Dom}(f) = \{x \in X \,|\, f(x) = y \text{ for some } y \in Y\} = X.$$

The *range* of f is

$$\mathrm{Ran}(f) = \{y \in Y \,|\, f(x) = y \text{ for some } x \in X\} \subset Y.$$

The *codomain* of f is $\mathrm{Codom}(f) = Y$.

Remarks.

1. The domain of *every* $f : X \to Y$ is *always* X.
2. The codomain of *every* $f : X \to Y$ is *always* Y, however $\mathrm{Ran}(f) \neq Y$ in general.

Examples

1.' If $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d, e\}$, and

$$f = \{(1, b), (2, a), (3, e), (4, a)\},$$
$$g = \{(1, a), (2, d), (3, b), (4, c)\},$$

(as above) then $\text{Ran}(f) = \{a, b, e\}$ and $\text{Ran}(g) = \{a, b, c, d\}$.

3.' Recall $f : [0, 3] \to \mathbb{R}$ given by $f(x) = (x - 1)^2$.

Claim: $\text{Ran}(f) = [0, 4]$.

Proof. ($\subseteq$) Let $y \in \text{Ran}(f)$. Then there is an $x \in [0, 3]$ so that $y = f(x) = (x - 1)^2$. We have

$$0 \leq x \leq 3 \;\Rightarrow\; -1 \leq x - 1 \leq 2 \;\Rightarrow\; 0 \leq (x - 1)^2 \leq 4.$$

Hence $y \in [0, 4]$. Therefore $\text{Ran}(f) \subseteq [0, 4]$.

($\supseteq$) Let $y \in [0,4]$. Then $\sqrt{y} \in [0,2]$ so that $x = \sqrt{y} + 1 \in [0,3]$. Moreover,

$$f(x) = (x-1)^2 = ((\sqrt{y}+1)-1)^2 = y,$$

so that $y \in \text{Ran}(f)$. Thus $[0,4] \subseteq \text{Ran}(f)$.

4.' Recall the projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$, given by

$$\pi_1(a,b) = a \quad \text{and} \quad \pi_2(a,b) = b.$$

Since every element of A or B occurs as a coordinate in $A \times B$:

$$\text{Ran}(\pi_1) = A \quad \text{and} \quad \text{Ran}(\pi_2) = B.$$

(provided $A, B \neq \varnothing$)

Images and Preimages

Definition

Let $f : X \to Y$ be a function.

1. For $A \subset X$, the *image of A under f* is
$$f(A) = \{f(a) \mid a \in A\} \subset Y.$$

2. For $B \subset Y$, the *preimage of B under f* is
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subset X.$$

Remark. Note that
$$f(X) = \text{Ran}(f) \quad \text{and} \quad f^{-1}(Y) = f^{-1}(\text{Ran}(f)) = X.$$

Examples

Again consider $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d, e\}$, and the functions

$$f = \{(1, b), (2, a), (3, e), (4, a)\},$$
$$g = \{(1, a), (2, d), (3, b), (4, c)\}.$$

Some images:

$$\begin{array}{ccc} f(\{1\}) = \{b\}, & f(\{1,2\}) = \{a, b\} & f(\{1,2,4\}) = \{a, b\}, \\ g(\{1,2\}) = \{a, d\}, & g(\{3,4\}) = \{b, c\} & g(\{2,3,4\}) = \{b, c, d\}. \end{array}$$

Some preimages:

$$\begin{array}{ccc} f^{-1}(\{a, b\}) = \{1, 2, 4\}, & f^{-1}(\{e\}) = \{3\}, & f^{-1}(\{a, c\}) = \{2, 4\}, \\ f^{-1}(\{c, d\}) = \varnothing, & g^{-1}(\{a, b\}) = \{1, 3\}, & g^{-1}(\{c\}) = \{4\}, \\ g^{-1}(\{a, e\}) = \{1\}, & g^{-1}(\{e\}) = \varnothing. & \end{array}$$

Example

Now consider $f : [0,3] \to \mathbb{R}$ given by $f(x) = (x-1)^2$.

If $x \in [0,1]$, then $-1 \leq x - 1 \leq 0$.

Thus $f(x) = (x-1)^2 \in [0,1]$, so that $f([0,1]) \subset [0,1]$.

Conversely, if $y \in [0,1]$, then $1 - \sqrt{y} \in [0,1]$ and

$$y = f(1 - \sqrt{y}) \in f([0,1]).$$

Hence $[0,1] \subset f([0,1])$.

Thus:

$$f([0,1]) = [0,1].$$

Example (Cont.)

On the other hand, we claim that $f^{-1}([0,1]) = [0,2]$.

Let $x \in [0,2]$. Then $-1 \leq x - 1 \leq 1$ so that

$$f(x) = (x-1)^2 \in [0,1] \implies x \in f^{-1}([0,1]) \implies [0,2] \subseteq f^{-1}([0,1]).$$

Now let $x \in f^{-1}([0,1])$. Then $f(x) = (x-1)^2 \in [0,1]$.

This implies $|x-1| = \sqrt{(x-1)^2} \leq 1$. Hence $-1 \leq x - 1 \leq 1$.

Therefore $0 \leq x \leq 2$ or $x \in [0,2]$. Thus $f^{-1}([0,1]) \subseteq [0,2]$

Having established double-containment, we conclude that

$$f^{-1}([0,1]) = [0,2]$$

Properties of Images and Preimages

Theorem 1

Let $f : X \to Y$ be a function, $A \subset X$ and $B \subset Y$. Then:

1. $f(f^{-1}(B)) \subset B$;
2. $A \subset f^{-1}(f(A))$.

Proof. **1.** Let $y \in f(f^{-1}(B))$. Then $y = f(x)$ for some $x \in f^{-1}(B)$.

But this means $y = f(x) \in B$. Hence $f(f^{-1}(B)) \subset B$.

2. Let $x \in A$. Then $f(x) \in f(A)$.

This is equivalent to $x \in f^{-1}(f(A))$. Hence $A \subset f^{-1}(f(A))$. $\square$

Remark

The containments of Theorem 1 *can be proper.*

Let $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d, e\}$, and

$$f = \{(1, b), (2, a), (3, e), (4, a)\}.$$

Then

$$f(f^{-1}(\{a, d\})) = f(\{2, 4\}) = \{a\} \subsetneq \{a, d\}$$

and

$$f^{-1}(f(\{2\})) = f^{-1}(\{a\}) = \{2, 4\} \supsetneq \{2\}.$$

Theorem 2

Let $f : X \to Y$ be a function, $A, B \subset X$, and $C, D \subset Y$. Then:

1. $f(A \cup B) = f(A) \cup f(B)$;
2. $f(A \cap B) \subset f(A) \cap f(B)$;
3. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$;
4. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Proof (sketch). We leave **1** and **3** as exercises.

2. Let $y \in f(A \cap B)$. Then there is an $x \in A \cap B$ so that $y = f(x)$.

Since $x \in A$, $y = f(x) \in f(A)$. Since $x \in B$, $y = f(x) \in f(B)$.

Thus $y = f(x) \in f(A) \cap f(B)$. Hence $f(A \cap B) \subset f(A) \cap f(B)$.

4. Let $x \in f^{-1}(C \cap D)$. Then $f(x) \in C \cap D$.

Since $f(x) \in C$, $x \in f^{-1}(C)$. Since $f(x) \in D$, $x \in f^{-1}(D)$.

Thus $x \in f^{-1}(C) \cap f^{-1}(D)$. Hence $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$.

Now let $x \in f^{-1}(C) \cap f^{-1}(D)$.

Since $x \in f^{-1}(C)$, $f(x) \in C$. Since $x \in f^{-1}(D)$, $f(x) \in D$.

Therefore $f(x) \in C \cap D$, so that $x \in f^{-1}(C \cap D)$.

Hence $f^{-1}(C) \cap f^{-1}(D) \subset f^{-1}(C \cap D)$, as well. $\square$

Remark

The containment of part **2** *can be proper*.

Again consider $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d, e\}$, and

$$f = \{(1, b), (2, a), (3, e), (4, a)\}.$$

Let $A = \{2\}$ and $B = \{4\}$.

Then $A \cap B = \varnothing$ so that $f(A \cap B) = \varnothing$.

But $f(A) = f(B) = \{a\} \neq \varnothing$.