Cartesian Products of Sets

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Intro to Abstract Mathematics

Ordered Pairs

Definition

Let A and B be sets. Given $a \in A$ and $b \in B$, the object (a, b) is called an *ordered pair*, with *first coordinate* a and *second coordinate* b.

Remarks.

1. Two ordered pairs are equal if and only if both of their coordinates match. That is,

$$(a, b) = (c, d) \quad \text{iff} \quad a = c \text{ and } b = d.$$

2. Unlike two element sets, the order of the coordinates in an ordered pair *matters*:

$$\{1, 2\} = \{2, 1\} \quad \text{but} \quad (1, 2) \neq (2, 1).$$

Cartesian Products

Definition

Let A and B be sets. The *(Cartesian) product* of A and B is the set

$$A \times B = \{(a, b) \,|\, a \in A \text{ and } b \in B\}$$

of all ordered pairs (a, b) with $a \in A$ and $b \in B$.

Remarks.

1. We write $A \times A = A^2$.
2. $\varnothing \times B = A \times \varnothing = \varnothing$.
3. $A \times B \neq B \times A$ unless $A = \varnothing$, $B = \varnothing$ or $A = B$.

Examples

- If $A = \{1, 2, 3\}$ and $B = \{x, y\}$, then

$$A \times B = \{(1, x), (2, x), (3, x), (1, y), (2, y), (3, y)\}.$$

- $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N} =$

$$\left\{\begin{matrix} (0,0), & (0,1), & (0,2), & (0,3), & \dots \\ (1,0), & (1,1), & (1,2), & (1,3), & \dots \\ (2,0), & (2,1), & (2,2), & (2,3), & \dots \\ (3,0), & (3,1), & (3,2), & (3,3), & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}\right\}$$

- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ is the *Cartesian plane*.

Properties of Products

Let A, B, C, D be sets.

1. $A \times B \subset C \times D$ iff $A \subset C$ and $B \subset D$ (if $A \neq \varnothing$ and $B \neq \varnothing$)

2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

3. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

4. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

5. $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$

Proofs

1. Suppose $A \times B \subset C \times D$. Let $a \in A$ and $b \in B$.

Then $(a, b) \in A \times B$, and so $(a, b) \in C \times D$, by hypothesis.

Thus $a \in C$ and $b \in D$. Since a and b were arbitrary, $A \subset C$ and $B \subset D$.

The converse is left as an exercise. □

2. We have

$$\begin{aligned}(a, b) \in A \times (B \cap C) &\Leftrightarrow a \in A \text{ and } b \in B \cap C \\ &\Leftrightarrow a \in A \text{ and } b \in B \text{ and } b \in C \\ &\Leftrightarrow (a, b) \in A \times B \text{ and } (a, b) \in A \times C \\ &\Leftrightarrow (a, b) \in (A \times B) \cap (A \times C).\end{aligned}$$

□

Properties **3.–5.** are similar, and are left as exercises. □

Warning. Property **1.** *does not* say that *every* subset of $C \times D$ has the form $A \times B$.

For example, if $C = D = \{1, 2\}$, then $S = \{(1, 1), (2, 2)\} \subset C^2$, but $S \neq A \times B$ for any A, B.

Warning. Notice that property **5.** is *not* an equality.

For example, if $A = [1, 3]$, $B = [2, 5]$, $C = [2, 4]$, $D = [4, 6]$ (closed intervals in $\mathbb{R}$), then $A \cup C = [1, 4]$ and $B \cup D = [2, 6]$.

The sets $A \times B$, $C \times D$ and $(A \cup C) \times (B \cup D)$ are all rectangles in $\mathbb{R}^2$, and $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$ by inspection.