Relations

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Intro to Abstract Mathematics

Relations

Definition

Let A and B be sets. A *relation from A to B* is a subset $R \subset A \times B$.

Remarks.

1. If R is a relation from A to B and $(a, b) \in R$, we write aRb.
2. Although R can be completely arbitrary, we think of aRb as specifying a *relationship* between a and b.
3. If $R \subset A^2$, then we say R is a *relation on A*.

Examples

1. Let $A = \{1, 2, 3\}$, $B = \{x, y\}$. Then

$$R = \{(1, x), (1, y), (2, x), (3, y)\}$$

   is a relation from A to B. We have $1Rx$, $1Ry$, $2Rx$ and $3Ry$.

2. Let $L = \{(x, y) \in \mathbb{N}^2 \mid x < y\}$. Then L is a relation on $\mathbb{N}$ with, e.g., $2L3$, $0L7$, and $4L13$.

3. Let $E = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3 - x\}$. Then E is a relation on $\mathbb{R}$ with, e.g., $0E0$, $(\pm 1)E0$ and $2E(\pm\sqrt{6})$.

4. Let $C = \{(m, n) \in \mathbb{Z} \mid 7 \text{ divides } n - m\}$. Then C is a relation on $\mathbb{Z}$ with, e.g., $8C1$, $4C25$, $0C49$ and $5C(-2)$.

5. For any set A, let $i_A = \{(a, a) \mid a \in A\}$. Then i_A is a relation on A.

The Power Set of a Set

Our next examples require the notion of a *power set*.

Definition

Let A be a set. The *power set* of A is the set

$$\mathcal{P}(A) = \{B \mid B \subset A\},$$

the set whose *elements* are the *subsets* of A.

Remark. Note that $B \subset A$ iff $B \in \mathcal{P}(A)$.

Examples.

a. $\mathcal{P}(\varnothing) = \{\varnothing\}$.

b. $\mathcal{P}(\{x, y\}) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}$.

c. R is a relation from A to B iff $R \in \mathcal{P}(A \times B)$.

d. The sets $\{7k+3 \mid k \in \mathbb{N}\}$, $\{n \in \mathbb{N} \mid n \text{ is even}\}$ and $\{p \mid p \text{ is prime}\}$ are elements of $\mathcal{P}(\mathbb{N})$.

e. We will see that there are as many elements of $\mathcal{P}(\mathbb{N})$ as there are real numbers.

Remarks.

1. If $n \in \mathbb{N}$ and A has exactly n elements, then $\mathcal{P}(A)$ has exactly 2^n elements. Hence the name "power set."
2. Another notation for $\mathcal{P}(A)$ is 2^A.
3. $\mathcal{P}(A)$ is *always* has *more* elements than A, even when A is *infinite*.

Back to Relations

Examples (cont.). Let A be a set.

6. $M = \{(a, B) \in A \times \mathcal{P}(A) \mid a \in B\}$ is a relation from A to $\mathcal{P}(A)$.

7. $S = \{(B, C) \in \mathcal{P}(A)^2 \mid C \subset B\}$ is a relation on $\mathcal{P}(A)$.

Definition

Let R be a relation from A to B. The *domain* of R is

$$\mathrm{Dom}(R) = \{a \in A \mid \exists b \in B(aRb)\} \subset A.$$

The *range* of R is

$$\mathrm{Ran}(R) = \{b \in B \mid \exists a \in A(aRb)\} \subset B.$$

Back to Our Examples

1. For the relation $R = \{(1, x), (1, y), (2, x), (3, y)\}$:
$$\mathrm{Dom}(R) = \{1, 2, 3\} = A, \quad \mathrm{Ran}(R) = \{x, y\} = B.$$

2. For the relation $L = \{(x, y) \in \mathbb{N}^2 \,|\, x < y\}$:
$$\mathrm{Dom}(L) = \mathbb{N}, \quad \mathrm{Ran}(L) = \mathbb{N}^+.$$

3. For the relation $E = \{(x, y) \in \mathbb{R}^2 \,|\, y^2 = x^3 - x\}$:
$$\mathrm{Dom}(E) = \{x \in \mathbb{R} \,|\, x^3 - x \geq 0\} = [-1, 0] \cup [1, \infty), \quad \mathrm{Ran}(E) = \mathbb{R}.$$

4. For the relation $C = \{(m, n) \in \mathbb{Z}^2 \,|\, 7 \text{ divides } n - m\}$:
$$\mathrm{Dom}(C) = \mathrm{Ran}(C) = \mathbb{Z}.$$

Inverses and Compositions

Definition

Let R be a relation from A to B. The *inverse* of R is

$$R^{-1} = \{(b, a) \in B \times A \mid aRb\}.$$

If S is a relation from B to C, the *composition* of S and R is

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B(aRb \wedge bSc)\}.$$

Remarks.

1. The inverse of a relation from A to B is a relation from B to A. One has aRb iff $bR^{-1}a$.
2. The composition of a relation from A to B with a relation from B to C is a relation from A to C.

Examples Again

1. The inverse of $R = \{(1,x),(1,y),(2,x),(3,y)\}$ is
$$R^{-1} = \{(x,1),(x,2),(y,1),(y,3)\}.$$

2. The inverse of $L = \{(x,y) \in \mathbb{N}^2 \,|\, x < y\}$ is
$$L^{-1} = \{(x,y) \in \mathbb{N}^2 \,|\, x > y\}.$$

3. The inverse of $E = \{(x,y) \in \mathbb{R}^2 \,|\, y^2 = x^3 - x\}$ is
$$E^{-1} = \{(x,y) \,|\, x^2 = y^3 - y\}.$$

4. The inverse of $C = \{(m,n) \in \mathbb{Z}^2 \,|\, 7 \text{ divides } n - m\}$ is
$$C^{-1} = \{(m,n) \,|\, 7 \text{ divides } m - n\} = C.$$

Some Compositions

Example 1

Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. Let

$$R = \{(1, 4), (1, 5), (2, 5), (3, 6)\}$$

be a relation from A to B, and let

$$S = \{(4, 5), (4, 6), (5, 4), (6, 6)\}$$

be a relation on B. Compute $S \circ R$ and $S \circ S^{-1}$.

Solution. Since $1R4$, $4S5$ and $4S6$, we have $(1, 5), (1, 6) \in S \circ R$.

Since $1R5$, $2R5$ and $5S4$, we find that $(1, 4), (2, 4) \in S \circ R$.

Since $3R6$ and $6S6$, we see that $(3, 6) \in S \circ R$.

It follows that

$$S \circ R = \{(1,4),(1,5),(1,6),(2,4),(3,6)\}.$$

Clearly

$$S^{-1} = \{(5,4),(6,4),(4,5),(6,6)\}.$$

Since $4S^{-1}5$ and $5S4$, we have $(4,4) \in S \circ S^{-1}$.

Since $5S^{-1}4$, $6S^{-1}4$ and $4S5$, $4S6$, we have

$$(5,5),(5,6),(6,5),(6,6) \in S \circ S^{-1}.$$

Since $6S^{-1}6$ and $6S6$, we have $(6,6) \in S \circ S^{-1}$. Thus

$$S \circ S^{-1} = \{(4,4),(5,5),(5,6),(6,5),(6,6)\}.$$

□

Example 2

Let $F = \{(x, x^2) \,|\, x \in \mathbb{R}\}$ and $G = \{(x, x+3) \,|\, x \in \mathbb{R}\}$ be relations on $\mathbb{R}$. Determine $F \circ G$ and $G \circ F$.

Solution. Let $x \in \mathbb{R}$. Then xFx^2. Furthermore, since $x^2 \in \mathbb{R}$, we have $x^2G(x^2+3)$. Therefore $(x, x^2+3) \in G \circ F$.

Conversely, if $(x, y) \in G \circ F$, then there is a $z \in \mathbb{R}$ so that xFz and zGy. Thus $z = x^2$ and $y = z + 3$, so that $y = x^2 + 3$. That is, $(x, y) = (x, x^2 + 3)$.

It follows that $G \circ F = \{(x, x^2+3) \,|\, x \in \mathbb{R}\}$.

Similarly one finds that $F \circ G = \{(x, (x+3)^2) \,|\, x \in \mathbb{R}\}$. □

Properties

The domain, range, inverse, and composition interact as follows.

Theorem 1

Let R be a relation from A to B, let S be a relation from B to C, and let T be a relation from C to D. Then:

1. $(R^{-1})^{-1} = R$.
2. $\mathrm{Dom}(R^{-1}) = \mathrm{Ran}(R)$.
3. $\mathrm{Ran}(R^{-1}) = \mathrm{Dom}(R)$.
4. $T \circ (S \circ R) = (T \circ S) \circ R$.
5. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Proof. We prove part **5** only. See the text for the rest.

Proof of Part 5

Note that both $(S \circ R)^{-1}$ and $R^{-1} \circ S^{-1}$ are relations from C to A.

Let $(c, a) \in (S \circ R)^{-1}$. Then $(a, c) \in S \circ R$.

Thus there exists $b \in B$ so that aRb and bSc.

Then $cS^{-1}b$ and $bR^{-1}a$, so that $(c, a) \in R^{-1} \circ S^{-1}$.

This proves that $(S \circ R)^{-1} \subset R^{-1} \circ S^{-1}$.

The opposite containment follows by reversing these steps. □