Strong Induction

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Intro to Abstract Mathematics

Motivating Example

Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ of integers defined by $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 5a_n - 6a_{n-1}$ for $n \geq 1$.

We say that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is defined *recursively*: any given term is determined by (the two) terms before it.

The first few terms of the sequence are

$$0, 1, 5, 19, 65, 211, 665, \ldots$$

In general, to compute a_n recursively for a given n, one must first compute $a_0, a_1, a_2, a_3, \ldots .a_{n-1}$.

Question 1: Can we find an *explicit* formula for a_n in terms of n alone? That is, can we express a_n in *closed form*?

Answer 1: Yes! We claim that $a_n = 3^n - 2^n$ for all $n \in \mathbb{N}$.

Indeed, for example we have

$$\begin{aligned}
3^0 - 2^0 &= 1 - 1 = 0 = a_0, \\
3^1 - 2^1 &= 3 - 2 = 1 = a_1, \\
3^2 - 2^2 &= 9 - 4 = 5 = a_2, \\
3^3 - 2^3 &= 27 - 8 = 19 = a_3, \\
3^4 - 2^4 &= 81 - 16 = 65 = a_4, \\
3^5 - 2^5 &= 243 - 32 = 211 = a_5.
\end{aligned}$$

Question 2: How can we prove this formula in general?

Answer 2: The recursive relationship $a_{n+1} = 5a_n - 6a_{n-1}$ makes induction a clear choice.

Missing Information

We just established 5 base cases, so let's look at the inductive step.

Suppose $a_n = 3^n - 2^n$ for some $n \geq 0$. Then

$$a_{n+1} = 5a_n - 6a_{n-1} = 5(3^n - 2^n) - 6\underbrace{a_{n-1}}_{?} = \cdots ?$$

The inductive hypothesis says nothing about a_{n-1}. What now?

If we also knew that $a_{n-1} = 3^{n-1} - 2^{n-1}$, then we'd have

$$\begin{aligned} 5(3^n - 2^n) - 6a_{n-1} &= 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) \\ &= 5 \cdot 3^n - 2 \cdot 3 \cdot 3^{n-1} - 5 \cdot 2^n + 3 \cdot 2 \cdot 2^{n-1} \end{aligned}$$

$$= 5 \cdot 3^n - 2 \cdot 3^n - 5 \cdot 2^n + 3 \cdot 2^n$$
$$= 3 \cdot 3^n - 2 \cdot 2^n = 3^{n+1} - 2^{n+1},$$

which proves the $n + 1$ case.

So the induction works provided we can take *two* previous cases as our inductive hypothesis.

This brings us to a weak form of strong induction known as *Recursive Induction.*

Recursive Induction allows one to assume any fixed number $k \geq 1$ of previous cases in the inductive hypothesis.

Recursive Induction

Theorem 1 (Recursive Induction)

Let $P(n)$ be a statement in the free variable n. Suppose there is a $k \in \mathbb{N}^+$ so that:

1. $P(0), P(1), P(2), \ldots, P(k-1)$ *are all true;*
2. *For all* $n \geq k-1$,

$$\left[P(n) \wedge P(n-1) \wedge \cdots \wedge P(n-k+1)\right] \Rightarrow P(n+1).$$

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Remarks.

1. This says we can use k previous cases when we induct, provided we check k base cases.
2. Recursive induction and standard induction are logically equivalent.

Back to the Sequence

Let's finish off our example.

Example 1

Define a sequence $\{a_n\}$ by $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 5a_n - 6a_{n-1}$ for $n \geq 1$. Prove that $a_n = 3^n - 2^n$ for all $n \in \mathbb{N}$.

Solution. We use (recursive) induction on $n \geq 0$ (with $k = 2$).

When $n = 0$ we have $a_0 = 0 = 3^0 - 2^0$, so the formula in question holds.

When $n = 1$ we have $a_1 = 1 = 3^1 - 2^1$, so the formula continues to hold.

Now let $n \geq 1$. Suppose $a_n = 3^n - 2^n$ and $a_{n-1} = 3^{n-1} - 2^{n-1}$.

Since $n \geq 1$, we know that

$$\begin{aligned} a_{n+1} &= 5a_n - 6a_{n-1} \\ &= 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) \\ &\vdots \text{ (as above)} \\ &= 3^{n+1} - 2^{n+1}, \end{aligned}$$

which is the $n + 1$ case of the formula.

By (recursive) induction, the formula holds for all $n \geq 0$. □

Remark. We always start the inductive step with "Let $n \geq \ell$," where $n = \ell$ is the final base case.

Another Motivating Example

Consider the following well-known result.

Example 2

Every integer $n \geq 2$ is either prime or a product of primes.

Remark. Here "prime" means "prime number."

Intuitively, if n isn't prime, we can factor it as $n = ab$ with $a, b > 1$.

If they aren't prime, factor a and b in the same way. Then factor their factors, etc.

Stop when all the factors become prime, as they must since indefinite factorization is impossible.

We can give a "clean" version of this argument using induction.

Let $P(n) =$ "n is prime or a product of primes."

Since 2 is prime, $P(2)$ is true, and we have at least one base case.

Let $n \geq 2$. We need to suppose $P(m)$ for some $2 \leq m < n+1$ (the inductive hypothesis) and conclude $P(n+1)$ is true.

Now $n+1$ is either prime or composite.

If $n+1$ is prime, $P(n+1)$ is true automatically. So suppose $n+1$ is composite.

Then $n+1 = ab$ with $1 < a, b < n+1$.

Which Hypotheses?

If $P(a)$ and $P(b)$ are true, then a, b are both primes or products of primes.

Thus $n+1 = ab$ is a product of primes, and $P(n+1)$ is true.

Question: In order for this argument to work, which $P(m)$ do we need to include in the inductive hypothesis?

All we know about the divisors a, b of $n+1$ is $1 < a, b < n+1$.

So we get "greedy" and suppose...

Answer: *All* $P(m)$ for $1 < m < n+1$.

This is known as *strong induction*.

Strong Induction

Theorem 2 (Strong Induction)

Let $P(n)$ be a statement in the free variable n. Suppose that:

1. $P(0)$ *is true;*
2. *For all* $n \in \mathbb{N}$, $\left[P(n) \wedge P(n-1) \wedge \cdots \wedge P(0)\right] \Rightarrow P(n+1)$.

Then $P(n)$ *is true for all* $n \in \mathbb{N}$.

Remarks.

1. Rather than say "Suppose $P(n)$ and $P(n-1)$ and ...," one simply says "Suppose $P(k)$ for all $0 \leq k \leq n$."
2. Strong induction is also logically equivalent to standard induction.

Back to Example 2

Let's return to our previous example.

Example 2

Every integer $n \geq 2$ is either prime or a product of primes.

Solution. We use (strong) induction on $n \geq 2$.

When $n = 2$ the conclusion holds, since 2 is prime.

Let $n \geq 2$ and suppose that for all $2 \leq k \leq n$, k is either prime or a product of primes.

Either $n + 1$ is prime or $n + 1 = ab$ with $2 \leq a, b, \leq n$.

In the latter case, the inductive hypothesis implies that a, b are primes or products of primes.

Then $n + 1 = ab$ is a product of primes.

So $n + 1$ is either prime or a product of primes, as needed.

By (strong) induction, the conclusion holds for all $n \geq 2$. $\square$

Remark. Note that although our inductive hypothesis is stronger than in recursive induction, we still only need a single base case.