Associativity of the Symmetric Difference

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Given sets A and B, their *symmetric difference* is

$$\begin{aligned} A\Delta B &= (A \setminus B) \cup (B \setminus A) && (1)\\ &= (A \cup B) \setminus (A \cap B). && (2) \end{aligned}$$

Because (1) (and (2)) is symmetric in A and B, we immediately find that Δ is commutative. That is, $A\Delta B = B\Delta A$. The purpose of this note is to prove the following less obvious property of the Δ operation.

Proposition 1. *The symmetric difference is associative. That is, given sets A, B and C, one has*

$$(A\Delta B)\Delta C = A\Delta(B\Delta C).$$

This proposition is an almost immediate consequence of the characterization of $(A\Delta B)\Delta C$ given below.

Lemma 1. *Let A, B and C be sets. Then*

$$(A\Delta B)\Delta C = \Big((A \cup B \cup C) \setminus [(A \cap B) \cup (A \cap C) \cup (B \cap C)]\Big) \cup (A \cap B \cap C). \tag{3}$$

To see how the proposition follows from the lemma, note that the right hand side of (3) is invariant under permutation of A, B and C. Thus

$$(A\Delta B)\Delta C = (B\Delta C)\Delta A = A\Delta(B\Delta C),$$

where we have used the commutativity of Δ to obtain the final equality. So all we need to do now is prove the lemma.

Proof of Lemma 1. By (1), (2) and the distributive laws for $\wedge$ and $\vee$ we have[1]

$$\begin{aligned} &x \in (A\Delta B)\Delta C \\ &\qquad \cong (x \in A\Delta B \wedge x \notin C) \vee (x \notin A\Delta B \wedge x \in C) \\ &\qquad \cong \Big([(x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)] \wedge x \notin C\Big) \vee \Big([x \notin A \cup B \vee x \in A \cap B] \wedge x \in C\Big) \\ &\qquad \cong [x \in A \wedge x \notin B \wedge x \notin C] \vee [x \notin A \wedge x \in B \wedge x \notin C] \vee [x \notin A \cup B \wedge x \in C] \vee [x \in A \cap B \wedge x \in C] \\ &\qquad \cong [x \in A \wedge x \notin B \wedge x \notin C] \vee [x \notin A \wedge x \in B \wedge x \notin C] \vee [x \notin A \wedge x \notin B \wedge x \in C] \vee [x \in A \cap B \cap C] \\ &\qquad \cong [x \in A \wedge x \notin B \cup C] \vee [x \in B \wedge x \notin A \cup C] \vee [x \in C \wedge x \notin A \cup B] \vee [x \in A \cap B \cap C]. \end{aligned}$$

For arbitrary statements P and Q, notice that

$$(P \vee Q) \wedge \neg Q \cong (P \wedge \neg Q) \vee (Q \wedge \neg Q) \cong P \wedge \neg Q,$$

[1] Because it is symmetric in A, B and C, the final statement in this chain of equivalences is sufficient to prove Proposition 1. It needs to be manipulated further, however, to prove Lemma 1.

since $Q \wedge \neg Q$ is a contradiction. Taking $P = (x \in A)$ and $Q = (x \in B \cup C)$, this implies

$$x \in A \wedge x \notin B \cup C \cong (x \in A \vee x \in B \cup C) \wedge x \notin B \cup C \cong x \in A \cup B \cup C \wedge x \notin B \cup C.$$

Similarly we have

$$\begin{aligned} x \in B \wedge x \notin A \cup C &\cong x \in A \cup B \cup C \wedge x \notin A \cup C, \\ x \in C \wedge x \notin A \cup B &\cong x \in A \cup B \cup C \wedge x \notin A \cup B. \end{aligned}$$

So our computation of $x \in (A \Delta B) \Delta C$ continues with

$$\begin{aligned} &\cong [x \in A \cup B \cup C \wedge x \notin B \cup C] \vee [x \in A \cup B \cup C \wedge x \notin A \cup C] \vee [x \in A \cup B \cup C \wedge x \notin A \cup B] \\ &\qquad \vee [x \in A \cap B \cap C] \\ &\cong \Big([x \in A \cup B \cup C] \wedge [x \notin B \cup C \vee x \notin A \cup C \vee x \notin A \cup B] \Big) \vee [x \in A \cap B \cap C] \\ &\cong \Big([x \in A \cup B \cup C] \wedge [x \notin (B \cup C) \cap (A \cup C) \cap (A \cup B)] \Big) \vee [x \in A \cap B \cap C] \\ &\cong x \in \Big((A \cup B \cup C) \setminus [(A \cup B) \cap (A \cup C) \cap (B \cup C)] \Big) \cup (A \cap B \cap C). \end{aligned}$$

The result now follows from the fact that

$$\begin{aligned} (A \cup B) \cap (A \cup C) \cap (B \cup C) &= [A \cup (B \cap C)] \cap (B \cup C) \\ &= [A \cap (B \cup C)] \cup [(B \cap C) \cap (B \cup C)] \\ &= (A \cap B) \cup (A \cap C) \cup (B \cap C), \end{aligned}$$

where in the final line we have used the fact that $(B \cap C) \cap (B \cup C) = B \cap C$ since $B \cap C \subset B \cup C$. □

Appendix

A simpler but more sophisticated proof can be given using *characteristic functions* of sets. Given a set A, its characteristic function χ_A is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Therefore $A = B$ if and only if $\chi_A = \chi_B$. Multiplication and addition of characteristic functions correspond to certain set operations. Specifically, $\chi_A \chi_B = \chi_{A \cap B}$ and, provided we add modulo 2 (i.e. we assert that 1 + 1 = 0), $\chi_A + \chi_B = \chi_{A \Delta B}$. Because addition modulo 2 is associative, we have

$$\begin{aligned} \chi_{(A \Delta B) \Delta C} &= \chi_{A \Delta B} + \chi_C \\ &= (\chi_A + \chi_B) + \chi_C \\ &= \chi_A + (\chi_B + \chi_C) \\ &= \chi_A + \chi_{B \Delta C} \\ &= \chi_{A \Delta (B \Delta C)}, \end{aligned}$$

and hence $(A \Delta B) \Delta C = A \Delta (B \Delta C)$.