Complex Variables
Spring 2020

Assignment 2.2
Due February 5

Exercise 1. Let $n \in \mathbb{N}$.

a. Explain why $X^n - 1 = \prod_{\zeta \in \mu_n} (X - \zeta)$.

b. Use part **a** to prove that $\prod_{\zeta \in \mu_n} \zeta = (-1)^{n+1}$ and $\sum_{\zeta \in \mu_n} \zeta = 0$ (for $n \geq 2$).

Exercise 2. Prove that for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ one has

$$\sum_{\zeta \in \mu_n} \zeta^k = \begin{cases} n, & \text{if } k \equiv 0 \pmod{n}, \\ 0, & \text{otherwise.} \end{cases}$$

[*Suggestion:* Use the fact that $\mu_n = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$, where $\omega = e^{2\pi i/n}$.]

Exercise 3. For $n \in \mathbb{N}$, let $f(X) = X^n - 1$ and $g(X) = X^{n-1} + X^{n-2} + \cdots + X + 1$ so that $f(X) = (X - 1)g(X)$. Suppose $\zeta \in \mu_n$, $\zeta \neq 1$.

a. Prove that $f'(\zeta) = (\zeta - 1)g'(\zeta)$. Conclude that $g'(\zeta) = \dfrac{n\zeta^{n-1}}{\zeta - 1}$.

b. Use part **a** to show that $n\zeta^{n-1} + (n-1)\zeta^{n-2} + \cdots + 2\zeta + 1 = \dfrac{n}{\zeta - 1}$.

Exercise 4. Compute μ_8.

Exercise 5. Find the 3rd roots of -8.

Exercise 6. Find the 4th roots of $a = 3 - 4i$. [*Warning:* Double check the sign of the imaginary part of your answer.]