Exercise 1. In this exercise you will prove the following strengthening of Cauchy's theorem for rectangles.

Theorem 1. *Let* $\Omega \subset \mathbb{C}$ *be a domain and* $z_0 \in \Omega$*. Suppose* $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ *is analytic and satisfies* $\lim_{z \to z_0}(z - z_0)f(z) = 0$*. If* R *is a closed rectangular region in* Ω *and* $z_0 \notin \partial R$*, then*

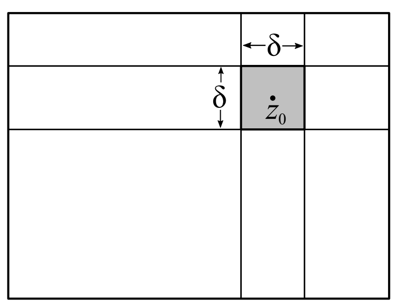
$$\int_{\partial R} f(z)\, dz = 0.$$

a. Explain why it suffices to assume z_0 is in the interior of R. [*Suggestion:* Appeal to the original version of Cauchy's theorem for rectangles.]

b. Subdivide R as shown below. Let R_0 denote the shaded rectangle. Explain why

$$\int_{\partial R} f(z)\, dz = \int_{\partial R_0} f(z)\, dz.$$

[*Suggestion:* See the previous suggestion.]



c. Let $\epsilon > 0$ and choose $\delta > 0$ so that $0 < |z - z_0| < \delta$ implies $|(z - z_0)f(z)| < \epsilon$. Show that $|f(z)| < 2\epsilon/\delta$ along ∂R_0.

d. Use an ML estimate to show that

$$\left| \int_{\partial R_0} f(z)\, dz \right| < 8\epsilon,$$

and use this to finish the proof.