

On the Differentiability of Certain Functions Defined by Path Integrals

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Let γ be a piecewise C^1 path in \mathbb{C} , suppose $g(z)$ is continuous on γ , and let $n \in \mathbb{N}$. Define $f : \mathbb{C} \setminus \gamma \rightarrow \mathbb{C}$ by

$$f(w) = \int_{\gamma} \frac{g(z)}{(z-w)^n} dz.$$

The purpose of this note is to provide a direct proof of the following result.

Theorem 1. *The function f is analytic on $\mathbb{C} \setminus \gamma$ and satisfies*

$$f'(w) = \frac{d}{dw} \int_{\gamma} \frac{g(z)}{(z-w)^n} dz = \int_{\gamma} \frac{\partial}{\partial w} \frac{g(z)}{(z-w)^n} dz = n \int_{\gamma} \frac{g(z)}{(z-w)^{n+1}} dz.$$

Proof. Fix $z_0 \in \mathbb{C} \setminus \gamma$. For $w \in \mathbb{C} \setminus \gamma$ we have

$$\begin{aligned} \frac{f(w) - f(z_0)}{w - z_0} &= \frac{1}{w - z_0} \int_{\gamma} g(z) \left(\frac{1}{(z-w)^n} - \frac{1}{(z-z_0)^n} \right) dz \\ &= \frac{1}{w - z_0} \int_{\gamma} g(z) \left(\frac{(z-z_0)^n - (z-w)^n}{(z-w)^n (z-z_0)^n} \right) dz \\ &= \frac{1}{w - z_0} \int_{\gamma} g(z) \left(\frac{((z-z_0) - (z-w)) \sum_{k=0}^{n-1} (z-z_0)^k (z-w)^{n-1-k}}{(z-w)^n (z-z_0)^n} \right) dz \\ &= \int_{\gamma} g(z) \sum_{k=0}^{n-1} \frac{1}{(z-z_0)^{n-k} (z-w)^{k+1}} dz \\ &= \int_{\gamma} g(z) \sum_{k=1}^n \frac{1}{(z-z_0)^{n-k+1} (z-w)^k} dz \end{aligned}$$

Therefore

$$\begin{aligned} \frac{f(w) - f(z_0)}{w - z_0} - n \int_{\gamma} \frac{g(z)}{(z-z_0)^{n+1}} dz &= \int_{\gamma} g(z) \sum_{k=1}^n \left(\frac{1}{(z-z_0)^{n-k+1} (z-w)^k} - \frac{1}{(z-z_0)^{n+1}} \right) dz \\ &= \sum_{k=1}^n \int_{\gamma} g(z) \frac{(z-z_0)^k - (z-w)^k}{(z-z_0)^{n+1} (z-w)^k} dz \\ &= \sum_{k=1}^n \int_{\gamma} g(z) \frac{((z-z_0) - (z-w)) \sum_{\ell=0}^{k-1} (z-z_0)^{k-1-\ell} (z-w)^{\ell}}{(z-z_0)^{n+1} (z-w)^k} dz \\ &= (w - z_0) \sum_{k=1}^n \sum_{\ell=0}^{k-1} \int_{\gamma} \frac{g(z)}{(z-z_0)^{n-k+\ell+2} (z-w)^{k-\ell}} dz \end{aligned}$$

Now let $R = \min\{|z - z_0| : z \in \gamma\} > 0$. Then $|z - z_0| \geq R$ for all $z \in \gamma$. Moreover, if $|w - z_0| < R/2$, then for any $z \in \gamma$ we have

$$|z - w| = |z - z_0 + z_0 - w| \geq |z - z_0| - |z_0 - w| \geq R - \frac{R}{2} = \frac{R}{2}.$$

Thus, for $|w - z_0| < R/2$, along γ we have

$$\left| \frac{1}{(z - z_0)^{n-k+\ell+2}(z - w)^{k-\ell}} \right| \leq \frac{1}{R^{n-k+\ell+2}(R/2)^{k-\ell}} = \frac{2^{k-\ell}}{R^{n+2}}.$$

Because g is continuous and γ is compact, there is a constant $M > 0$ so that $|g(z)| \leq M$ for $z \in \gamma$. Our work above then implies

$$\begin{aligned} \left| \frac{f(w) - f(z_0)}{w - z_0} - n \int_{\gamma} \frac{g(z)}{(z - z_0)^{n+1}} dz \right| &\leq |w - z_0| \left| \sum_{k=1}^n \sum_{\ell=0}^{k-1} \int_{\gamma} \frac{g(z)}{(z - z_0)^{n-k+\ell+2}(z - w)^{k-\ell}} dz \right| \\ &\leq |w - z_0| \sum_{k=1}^n \sum_{\ell=0}^{k-1} \frac{2^{k-\ell} ML(\gamma)}{R^{n+2}} \\ &= |w - z_0| \frac{ML(\gamma)}{R^{n+2}} \sum_{k=1}^n \sum_{\ell=0}^{k-1} 2^{k-\ell} \\ &= |w - z_0| \frac{ML(\gamma)}{R^{n+2}} (2^{n+2} - 2n - 4). \end{aligned}$$

This can be made arbitrarily small by taking w sufficiently close to z_0 , proving that

$$f'(z_0) = \lim_{w \rightarrow z_0} \frac{f(w) - f(z_0)}{w - z_0} = n \int_{\gamma} \frac{g(z)}{(z - z_0)^{n+1}} dz.$$

Since $z_0 \in \mathbb{C} \setminus \gamma$ was arbitrary, the statement of the theorem follows. \square

Notice that $f'(w)/n$ has the same form as $f(w)$, but with the exponent $n+1$ in place of n . Because $n \in \mathbb{N}$ was arbitrary in Theorem 1, we can apply the theorem in the $n+1$ case to conclude that $f'(w) = n \cdot f'(w)/n$ is analytic on $\mathbb{C} \setminus \gamma$, with

$$f''(w) = \frac{d}{dw} \left(n \int_{\gamma} \frac{g(z)}{(z - w)^{n+1}} dz \right) = n \int_{\gamma} \frac{\partial}{\partial w} \frac{g(z)}{(z - w)^{n+1}} dz = n(n+1) \int_{\gamma} \frac{g(z)}{(z - w)^{n+2}} dz.$$

An inductive argument can be used to show that this line of reasoning continues indefinitely.

Corollary 1. *f is infinitely differentiable on $\mathbb{C} \setminus \gamma$. For every $k \geq 0$,*

$$f^{(k)}(w) = \frac{(n+k-1)!}{(n-1)!} \int_{\gamma} \frac{g(z)}{(z - w)^{n+k}} dz.$$

In particular, differentiation under the integral sign is valid.