More on the Cauchy-Riemann Equations

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Complex Variables

The Set-Up

Throughout what follows:

- $x, y \in \mathbb{R}$, $z = x + iy$
- $\Omega \subset \mathbb{C}$ a domain
- $f : \Omega \to \mathbb{C}$ a function
- $u(x, y) = \operatorname{Re} f(z)$
- $v(x, y) = \operatorname{Im} f(z)$

Last Time

Theorem 1 (Cauchy-Riemann Equations)

Assume $u, v \in C^1(\Omega)$. Then $f = u + iv$ is analytic on Ω if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

and in this case

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} := \frac{\partial f}{\partial x} \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} := \frac{1}{i}\frac{\partial f}{\partial y}. \end{aligned}$$

Remark: The $\Rightarrow$ direction holds pointwise without C^1 hypothesis.

Partial Differential Operators

Formally viewing $z = x + iy$, $\overline{z} = x - iy$ as a change of variables, since $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$, a formal application of the chain rule yields

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

One can show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ are $\mathbb{C}$-linear and obey the product rule. Moreover, the C-R equations simply become

$$\frac{\partial f}{\partial \overline{z}} = 0,$$

with

$$\frac{\partial f}{\partial z} = f'(z).$$

The Meaning of $\partial/\partial\overline{z}$

In the form $\frac{\partial f}{\partial \overline{z}} = 0$, the C-R equations suggest that f is analytic provided it "doesn't depend on $\overline{z}$."

Example 1

For what $m, n \in \mathbb{N}_0$ is $f(x, y) = z^m \overline{z}^n$ analytic?

Solution. The function $z^m\overline{z}^n$ is a polynomial in x and y, so is $C^1(\mathbb{C})$. It therefore suffices to check the C-R equations. We first note that

$$\frac{\partial z}{\partial \overline{z}} = 0$$

because z is analytic, while

$$\frac{\partial \overline{z}}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}(x - iy) - \frac{1}{i}\frac{\partial}{\partial y}(x - iy)\right) = \frac{1}{2}(1 + 1) = 1,$$

as expected.

So, by the power rule (which is a consequence of the product rule),

$$\frac{\partial}{\partial \overline{z}} z^m \overline{z}^n = m z^{m-1} \frac{\partial z}{\partial \overline{z}} \overline{z}^n + z^m n \overline{z}^{n-1} \frac{\partial \overline{z}}{\partial \overline{z}} = n z^m \overline{z}^{n-1}.$$

This vanishes identically if and only if $\boxed{n = 0}$, i.e. $f(x, y) = z^m$. □

More generally, one can prove the following result.

Theorem 2

Let $P(x, y) \in \mathbb{C}[x, y]$ and set $Q(z, \overline{z}) = P\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) \in \mathbb{C}[z, \overline{z}]$. Then P is analytic if and only if there is an $f(z) \in \mathbb{C}[z]$ so that $P(x, y) = f(x + iy)$. That is, $\frac{\partial Q}{\partial \overline{z}} = 0$ if and only if Q has no monomials of the form $az^m\overline{z}^n$ with $a \neq 0$ and $n \geq 1$.

Examples

Example 2

The polynomial $x^2 + y^2$ *is* not *an analytic function since*

$$x^2 + y^2 = |z|^2 = z\overline{z}.$$

Example 3

Let $f(x, y) = x^2 + 2ixy - y^2$. *Then*

$$\begin{aligned} f\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) &= \frac{z^2 + 2z\overline{z} + \overline{z}^2}{4} + \frac{z^2 - \overline{z}^2}{2} - \frac{z^2 - 2z\overline{z} + \overline{z}^2}{-4} \\ &= z^2, \end{aligned}$$

so that f *is analytic everywhere (entire).*

The Jacobian

If we view f as a real map $(x, y) \mapsto (u(x, y), v(x, y))$, its *Jacobian* is the matrix

$$J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

The Jacobian is the analogue of the real-variable derivative in higher dimensions.

The C-R equations immediately imply the following classification of analytic functions in terms of their Jacobians.

Theorem 3

Let $u, v \in C^1(\Omega)$. Then f is analytic on Ω if and only if J_f represents a complex number at each point of Ω. That is, if and only if J_f has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ throughout Ω. In this case, $f' = a + ib$.

Linear Approximations

Let $u, v \in C^1(\Omega)$. The *linear approximation* of $f(x, y)$ at $(x_0, y_0) \in \Omega$ is given by

$$L(x, y) = f(x_0, y_0) + J_f(x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where we view points in $\mathbb{R}^2$ as 2×1 column vectors.

Notice that if J_f represents a complex number $a + ib$ and we view $x + iy$ as a column vector, then

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix},$$

which is just the column vector form of $(a + ib)(x + iy)$

We conclude that if f is analytic on Ω, then when viewed in $\mathbb{C}$ the linear approximation becomes

$$L(z) = f(z_0) + f'(z_0)(z - z_0).$$

This perfectly matches the formula for the linear approximation used in Calculus I!

Remark. One can use the real variable Inverse Function Theorem to study the inverses of analytic functions via their Jacobians. See section 2.4. We will postpone this until the introduction of power series.

The C-R Equations in Polar Coordinates

The multivariate chain rule can be used to express the C-R equations in terms of polar coordinates.

If

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$

then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,$$
$$\frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta.$$

and similarly for v.

In matrix form this system becomes

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

We can also express the C-R equations using matrices:

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}.$$

Substitution of the chain rule matrix equations from above yields the *polar Cauchy-Riemann* equations:

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial u}{\partial \theta}, \\ \frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r}. \end{aligned}$$

These can be used to test the analyticity of functions more easily expressed in polar coordinates.

Examples

Example 4

Let $f(z) = z^m$, $m \in \mathbb{N}$. Show that f is entire using the C-R equations.

Solution. The real and imaginary parts of $f(x+iy) = (x+iy)^m$ are complicated. In polar coordinates, however,

$$f(re^{i\theta}) = r^m e^{im\theta} = \underbrace{r^m \cos m\theta}_{u} + i\,\underbrace{r^m \sin m\theta}_{v}.$$

Thus

$$\begin{aligned}
\frac{\partial u}{\partial r} &= mr^{m-1}\cos m\theta = \frac{r^m m \cos m\theta}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\\
\frac{\partial u}{\partial \theta} &= -r^m m \sin m\theta = -r(mr^{m-1}\sin m\theta) = -r\frac{\partial v}{\partial r}.
\end{aligned}$$

Because $f \in C^1(\mathbb{C})$, we are finished.

Differentiability of the Logarithm

Recall that $\log z = \ln|z| + i\arg z$. In polar coordinates this is

$$\log re^{i\theta} = \ln r + i\theta.$$

This is in $C^1(\Omega)$ on any domain Ω in which θ varies continuously.

Moreover

$$\begin{aligned} \frac{\partial}{\partial r}\ln r &= \frac{1}{r} = \frac{1}{r}\frac{\partial\theta}{\partial\theta}, \\ \frac{\partial}{\partial\theta}\ln r &= 0 = -r\frac{\partial\theta}{\partial r}. \end{aligned}$$

It follows that $\log z$ is analytic anywhere $\arg z$ is continuous.

What about its derivative?

Differentiating the Logarithm

The derivative of (any branch of) $\log z$ is most easily computed using the chain rule.

Indeed, we have $z = \exp(\log z)$ for all z. So anywhere that $\log z$ is analytic the chain rule implies

$$1 = \frac{dz}{dz} = \exp(\log z) \cdot \frac{d}{dz} \log z = z \cdot \frac{d}{dz} \log z \quad \Rightarrow \quad \frac{d}{dz} \log z = \frac{1}{z}.$$

We have proven the following result.

Theorem 4

The logarithm $\log z = \ln |z| + i \arg z$ *is analytic on any domain* Ω *on which* $\arg z$ *is continuous, with derivative*

$$\frac{d}{dz} \log z = \frac{1}{z}.$$