Fractional Linear Transformations

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Complex Variables

Fractional Linear Transformations

We now consider a particular class of conformal maps.

Definition

A rational function of the form

$$f(z) = \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ is called a *fractional linear transformation* (FLT) (or *Möbius transformation*).

Notice

$$f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2},$$

so that FLTs are conformal where they are defined.

Example 1

Every linear function $f(z) = az + b$ is an FLT (with $c = 0$, $d = 1$). These include the *dilations* ($z \mapsto az$) and *translations* ($z \mapsto z + b$).

Example 2

The *inversion* $f(z) = 1/z$ is an FLT ($a = 0, b = c = 1, d = 0$).

If $c \neq 0$, then polynomial long division gives

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{(ad-bc)/c}{cz+d}.$$

Together with the $c = 0$ case we find that:

Theorem 1

Every FLT is a composition of dilations, translations and inversions.

FLTs and $\widehat{\mathbb{C}}$

Every FLT $f(z) = \frac{az+b}{cz+d}$ extends naturally to a map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

The two questions to address are:

1. How do we define $f(z)$ where $cz + d = 0$?
2. How do we define $f(\infty)$?

In both cases we will simply take a limit.

Recall that $z \to \infty$ in $\widehat{\mathbb{C}}$ provided $|z| \to \infty$ in $\mathbb{R}$.

The usual limit laws hold for limits to ∞. In particular,

$$\lim_{z\to\infty} \frac{1}{z^n} = 0$$

for $n \in \mathbb{N}$.

$c \neq 0$

Thus, if $c \neq 0$, then

$$\lim_{z\to\infty} \frac{az+b}{cz+d} = \lim_{z\to\infty} \frac{a+\frac{b}{z}}{c+\frac{d}{z}} = \frac{a+0}{c+0} = \frac{a}{c}.$$

For an FLT $f(z) = \frac{az+b}{cz+d}$ $(c \neq 0)$ we therefore define $f(\infty) = a/c$.

Since $cz + d$ vanishes when $z = -d/c$, $f(z)$ isn't defined there.

However,

$$\lim_{z\to -d/c} az + b = \frac{-ad}{c} + b = \frac{bc - ad}{c} \neq 0.$$

It follows that $\frac{az+b}{cz+d}$ can be made arbitrarily large as $z \to -d/c$.

So we set $f(-d/c) = \lim_{z\to -d/c} f(z) = \infty$.

$c = 0$

If $c = 0$, then $f(z) = az + b$ with $a \neq 0$ (WLOG), which is entire.

By the reverse triangle inequality

$$|f(z)| \geq \Big| |a||z| - |b| \Big| \geq |a||z| - |b|.$$

As $z \to \infty$, the RHS can be made arbitrarily large, i.e. $f(z) \to \infty$.

So for FLTs of the form $f(z) = az + b$ we define

$$f(\infty) = \lim_{z \to \infty} az + b = \infty.$$

Note that in every case, we have extended our FLTs to be *continuous throughout* $\widehat{\mathbb{C}}$.

Inverses

Every *FLT* $f(z) = \frac{az+b}{cz+d}$ on the extended plane $\widehat{\mathbb{C}}$ is invertible.

In fact, solving the equation $z = \frac{aw+b}{cw+d}$ for w yields

$$w = \frac{dz - b}{-cz + a}.$$

One can check that the FLT $g(z) = \frac{dz-b}{-cz+a}$ satisfies $g(f(z)) = f(g(z)) = z$ at every extended value as well, e.g.

$$f(g(a/c)) = f(\infty) = a/c$$

when $c \neq 0$.

So every FLT is invertible on $\widehat{\mathbb{C}}$, with inverse given by another FLT.

Composition

Because the identity function $f(z) = z$ is an FLT, and function composition is associative, we are one step away from showing the FLTs make up a group.

It remains to check a composition of FLTs is another FLT.

Let $f(z) = \frac{az+b}{cz+d}$ and $g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ be FLTs. Then

$$f(g(z)) = \frac{a(\alpha z+\beta)+b(\gamma z+\delta)}{c(\alpha z+\beta)+d(\gamma z+\delta)} = \frac{(a\alpha+b\gamma)z+(a\beta+b\delta)}{(c\alpha+d\gamma)z+(c\beta+d\delta)}.$$

This is an FLT since, as one can show

$$(a\alpha+b\gamma)(c\beta+d\delta)-(a\beta+b\delta)(c\alpha+d\gamma) = (ad-bc)(\alpha\delta-\beta\gamma) \neq 0.$$

The set of FLTs is therefore a group under function composition.

Group Structure

Theorem 2

The set $\mathcal{F}$ of fractional linear transformations is a group under composition of maps. $\mathcal{F}$ is generated by dilations, translations and inversion.

There is another (perhaps now obvious) group theoretic connection.

Given $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$, let $T_A(z) = \frac{az+b}{cz+d}$. Then $T_A \in \mathcal{F}$ and the computations of the preceding slide show that

$$T_A \circ T_B = T_{AB}.$$

That is, the map $\tau : \mathrm{GL}_2(\mathbb{C}) \to \mathcal{F}$ given by $A \mapsto T_A$ is a homomorphism, obviously surjective.

$\ker \tau$

T_A belongs to the kernel of τ if and only if T_A is the identity function $z \mapsto z$.

This means $T_A(z) = \frac{az+b}{cz+d} = z$ for all z, or

$$cz^2 + (d-a)z - b = 0$$

for all $z \in \mathbb{C}$.

A nonzero polynomial has only finitely many roots, so this occurs iff $b = c = 0$ and $a = d$, or $A = \left(\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right) = aI$, a *scalar matrix*.

So the kernel of τ consists precisely of the scalar matrices:

$$\ker \tau = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \,\middle|\, a \in \mathbb{C}^\times \right\} = \mathbb{C}^\times I.$$

The Projective General Linear Group

The first isomorphism theorem now yields:

Theorem 3

The map $A \mapsto T_A$ induces an isomorphism between the projective general linear group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times I$ *and the group $\mathcal{F}$ of fractional linear transformations of $\widehat{\mathbb{C}}$.*

Because elements in the same fiber differ by an element of the kernel, we have the following immediate consequence.

Corollary 1

Let $A, B \in \mathrm{GL}_2(\mathbb{C})$. Then $T_A = T_B$ if and only if there is a $\lambda \in \mathbb{C}^\times$ so that $A = \lambda B$.

This completely answers the question of uniqueness of coefficients in an FLT.

Fixed Points

Let $f(z) = \frac{az+b}{cz+d}$ be an FLT. A point $z_0 \in \widehat{\mathbb{C}}$ is a *fixed point* of f provided $f(z_0) = z_0$.

We will use fixed points to help us distinguish FLTs.

Since $f(\infty) = a/c$ if $c \neq 0$, we find that ∞ is a fixed point iff $c = 0$.

If $z_0 \in \mathbb{C}$, then $f(z_0) = z_0$ if and only if $cz_0^2 + (d-a)z_0 - b = 0$.

This has at most two solutions if $c \neq 0$, and at most one if $c = 0$, unless $a = d$ and $b = 0$, in which case f is trivial.

Theorem 4

A nonidentity FLT has at most two fixed points in $\widehat{\mathbb{C}}$.

Examples

Example 3

The only fixed point of the FLT $f(z) = z + 1$ is $z = \infty$. The same is true of all translations.

Example 4

The fixed points of $f(z) = 2z$ are $z = 0, \infty$. The same is true of all dilations.

Example 5

The fixed points of $f(z) = \frac{z+1}{z-1}$ are $z = 1 \pm \sqrt{2}$.

Example 6

The only fixed point of $f(z) = \frac{3z-1}{z+1}$ is $z = 1$.

Values of FLTs

Theorem 5

If two FLTs agree at three points in $\widehat{\mathbb{C}}$, they are identical.

Proof. Let $f, g \in \mathcal{F}$. Suppose $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ are distinct and that $f(z_j) = g(z_j)$ for $j = 1, 2, 3$.

Then z_1, z_2 and z_3 are fixed points of $g^{-1} \circ f$.

Thus $g^{-1} \circ f = I$ and hence $f = g$. □

So an FLT is determined by its values at any three points. To what extent can these be specified?

Lemma 1

Let $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ be distinct. There is a unique FLT carrying z_1, z_2, z_3 to $0, 1, \infty$, respectively.

Proof. Uniqueness is guaranteed by the theorem. To establish existence, simply take

$$f(z) = \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1}. \tag{1}$$

□

Remark. If one of z_1, z_2 or z_3 is ∞, simply take the appropriate limit in (1).

For example, holding z, z_2 and z_3 fixed, we find that

$$\lim_{z_1 \to \infty} \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1} = \frac{z_2 - z_3}{z - z_3},$$

which carries ∞, z_2, z_3 to $0, 1, \infty$.

The Cross-Ratio

Definition

If $z_0, z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ are distinct, their *cross-ratio* is

$$[z_0, z_1, z_2, z_3] = \frac{z_0 - z_1}{z_0 - z_3}\frac{z_2 - z_3}{z_2 - z_1},$$

which is the image of z_0 under the FLT carrying z_1, z_2, z_3 to $0, 1, \infty$.

We can use the cross-ratio to give a simple proof of the next result.

Theorem 6

Let $z_1, z_2, z_3, w_1, w_2, w_3 \in \widehat{\mathbb{C}}$. If the z_j are pairwise distinct and the w_j are pairwise distinct, then there exists a unique $f \in \mathcal{F}$ so that $f(z_j) = w_j$ for $j = 1, 2, 3$.

Invariance

Proof. Let $g(z) = [z, z_1, z_2, z_3]$ and $h(w) = [w, w_1, w_2, w_3]$.

Then $f = h^{-1} \circ g$ works, and is necessarily unique by Theorem 7. □

One consequence of Theorem 8 is the invariance of the cross-ratio under FLTs.

Theorem 7

Let $z_0, z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ be distinct. For any $f \in \mathcal{F}$,

$$[f(z_0), f(z_1), f(z_2), f(z_3)] = [z_1, z_2, z_3, z_4].$$

Proof. Let $g(z) = [z, z_1, z_2, z_3]$ and $h(w) = [w, f(z_1), f(z_2), f(z_3)]$.

Then $h^{-1} \circ g$ maps $z_j \mapsto f(z_j)$ for $j = 1, 2, 3$.

By Theorem 7, this implies $h^{-1} \circ g = f$ or $g = h \circ f$.

Thus

$$[f(z_0), f(z_1), f(z_2), f(z_3)] = h(f(z_0)) = g(z_0) = [z_0, z_1, z_2, z_3].$$

$\square$

Preservation of the cross-ratio is related to the following important geometric property of FLTs.

Theorem 8

A fractional linear transformation $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ carries circles to circles.

Remark. A *circle* in $\widehat{\mathbb{C}}$ is a circle or line (circle through ∞) in $\mathbb{C}$.

Proof

By Theorem 1 it suffices to assume f is a translation, dilation or inversion. The result is clear if f is a translation or dilation.

So assume $f(z) = 1/z$. Recall the stereographic projection $\pi : \widehat{\mathbb{C}} \to S^2$.

Let R denote rotation rotation about the X-axis in $\mathbb{R}^3$ by $180°$.

According to exercise I.3.4, R corresponds to the inversion map $z \mapsto 1/z$ on $\widehat{\mathbb{C}}$. That is,

$$R = \pi \circ f \circ \pi^{-1} \Leftrightarrow f = \pi^{-1} \circ R \circ \pi.$$

But π and R map circles to circles, so we're finished. $\square$

Application

Three distinct points $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ uniquely determine a circle C.

If $f(z)$ is an FLT, $f(C)$ is a circle containing the distinct points $f(z_1)$, $f(z_2)$, $f(z_3)$. There is only one such circle.

Moral. To compute the image of a circle under an FLT, one need only compute the image of any three of its points.

Example 7

Show that $f(z) = \frac{iz-1}{iz+1}$ maps the real axis to the unit circle, the unit circle to the imaginary axis, and the imaginary axis to the real axis.

Solution

One easily checks that under f:

$$-i \mapsto 0, \quad i \mapsto \infty, \quad \infty \mapsto 1, \quad 0 \mapsto -1, \quad 1 \mapsto i.$$

0, 1 and ∞ are on the real axis and map to -1, i, 1 on the unit circle. Thus $\mathbb{R} \mapsto S^1$.

1, i, $-i$ on the unit circle map to i, ∞, 0 on the imaginary axis. Hence $S^1 \mapsto i\mathbb{R}$.

$-i$, 0 and i on the imaginary axis map to 0, -1, ∞ on the real axis. So $i\mathbb{R} \mapsto \mathbb{R}$. □

Regions

Let $f(z)$ be an FLT, C be a circle in $\widehat{\mathbb{C}}$, and $C' = f(C)$.

C divides $\widehat{\mathbb{C}}$ into two connected components, Ω_1 and Ω_2 (the "inside" and "outside" of C).

Likewise, C' divides $\widehat{\mathbb{C}}$ into Ω_1' and Ω_2'.

Topological considerations imply that, after possibly relabeling, $f(\Omega_j) = \Omega_j'$, $j = 1, 2$.

Conformality implies f preserves the relative orientation of these "sides" to C.

Example 8

Show that $f(z) = \frac{iz-1}{iz+1}$ maps the upper half-plane $H = \{z \mid \operatorname{Im} z > 0\}$ to the complement D^c of the closed unit disk.

Solution. We have seen that f maps $0, 1, \infty$ on the real axis to $-1, i, 1$ on the unit circle.

So f must map H to either the "inside" or the "outside" of the unit circle.

As we move along $\mathbb{R}$ from 0 to 1 to ∞, H is on the left.

So as we move on the unit circle from -1 to i to 1, the image of H must also be on the left, or "outside" the unit circle.

Hence f maps H to D^c. □

Another Application

Finally, we can interpret the cross-ratio.

Theorem 9

Let $z_0, z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ be distinct. The cross-ratio $[z_0, z_1, z_2, z_3]$ is real if and only if z_0, z_1, z_2, z_3 lie on the same circle.

Proof. Let C be the circle through z_1, z_2, z_3 and $f(z) = [z, z_1, z_2, z_3]$.

Then $f \in \mathcal{F}$ carries

$$\begin{matrix} z_1, z_2, z_3 \\ \text{on } C \end{matrix} \mapsto \begin{matrix} 0, 1, \infty \\ \text{on } \mathbb{R}. \end{matrix}$$

Hence $f(C) = \widehat{\mathbb{R}}$.

So $f(z_0) \in \mathbb{R}$ if and only if $z_0 \in C \setminus \{z_3\}$. □