Power Series

Ryan C. Daileda

Trinity University

Complex Variables

Introduction

Definition

A *power series* (PS) *centered* at $z_0 \in \mathbb{C}$ is an expression of the form

$$\sum_{k=0}^{\infty} a_k(z - z_0)^k, \quad a_k \in \mathbb{C}. \tag{1}$$

The terms in the sequence $\{a_k\}$ are called the *coefficients* of (1).

Remarks.

1. PS can be viewed as infinite-degree generalizations of polynomials, and share many (though not all!) of their properties.
2. We can always viewed as a PS as centered at $z_0 = 0$ via the simple substitution $w = z - z_0$.

Example 1

The geometric series $\sum z^k$ is a power series centered at $z_0 = 0$. It converges (normally) for $|z| < 1$, and diverges otherwise.

We will see that all power series share similar convergence properties.

We begin with a fundamental lemma.

Lemma 1

If the PS $\sum a_k z^k$ converges at a nonzero point z_0, then it converges absolutely and normally on $|z| < |z_0|$.

Proof. Let $R = |z_0|$ and choose $0 < r < R$.

Because $\sum a_k z_0^k$ converges, $a_k z_0^k \to 0$.

In particular, $\left|a_k z_0^k\right| = |a_k| R^k < 1$ for all $k \geq K$.

If $|z| \leq r < R$, then for $k \geq K$ we have

$$\left|a_k z^k\right| = |a_k| \cdot |z|^k \leq |a_k| r^k = |a_k| R^k \left(\frac{r}{R}\right)^k < \left(\frac{r}{R}\right)^k.$$

Thus, $\sum |a_k z^k|$ is (eventually) dominated by the convergent geometric series $\sum (r/R)^k$.

By the M-test, the series $\sum a_k z^k$ converges absolutely and uniformly for $|z| \leq r$. □

Corollary 1

If the PS $\sum a_k z^k$ diverges at a nonzero point z_0, then it diverges for all $|z| > |z_0|$.

Proof. Suppose, instead, that $\sum a_k z^k$ converges for some $|z| > |z_0|$.

It then converges at z_0 by the Lemma, contradicting our hypothesis. □

Definition

Let $\sum a_k z^k$ be a power series and set

$$R = \sup\left\{|z| \,:\, \sum a_k z^k \text{ converges}\right\},$$

the *radius of convergence* of $\sum a_k z^k$.

Remark. It is possible to have $R = 0$ or $R = \infty$.

Convergence of Power Series

Theorem 1

The radius of convergence R of a PS $\sum a_k z^k$ is uniquely defined by the following properties:

1. $\sum a_k z^k$ *converges absolutely and normally for* $|z| < R$;
2. $\sum a_k z^k$ *diverges for* $|z| > R$.

Proof. By definition, $\sum a_k z^k$ diverges for $|z| > R$, so **2** holds.

As for **1**, let $0 < r < R$.

Because R is the supremum, there is a $|z_0| \leq R$ so that $r < |z_0|$ and $\sum a_k z_0^k$ converges.

For $|z| \leq r$, the conclusion follows from the fundamental Lemma.

(Uniqueness is an easy exercise.) □

Examples

Example 2

Determine the radius of convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k2^k}$.

Solution. We appeal to the root test for absolute convergence:

$$L = \lim_{k\to\infty} \sqrt[k]{\left|\frac{z^k}{k2^k}\right|} = \lim_{k\to\infty} \frac{|z|}{2\sqrt[k]{k}} = \frac{|z|}{2}$$

The series converges absolutely when $L < 1$ and diverges if $L > 1$.

It follows that the radius of convergence is $\boxed{R = 2}$. □

Example 3

Find the radius of convergence of the exponential series $\sum_{k=0}^{\infty} \frac{z^k}{k!}$.

Solution. We use the ratio test for absolute convergence:

$$L = \lim_{k\to\infty} \left| \frac{z^{k+1}}{(k+1)!} \bigg/ \frac{z^k}{k!} \right| = \lim_{k\to\infty} \frac{|z|}{k+1} = 0$$

for all $z \in \mathbb{C}$.

Because $L < 1$, the ratio test implies the series converges absolutely for all $z \in \mathbb{C}$.

Thus the radius of convergence is $\boxed{R = \infty}$. □

Differentiability of Power Series

Theorem 2

Let $\sum a_k z^k$ be a power series with positive radius of convergence R. The function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is analytic on $|z| < R$ and can be differentiated term-by-term, infinitely often:

$$f^{(m)}(z) = \frac{d^m}{dz^m} \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} \frac{d^m}{dz^m} a_k z^k = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m},$$

with absolute and normal convergence for $|z| < R$.

Proof

The terms in a power series are entire functions, and power series converge normally inside their radii of convergence.

The result is an immediate consequence of the theorem on normal convergence of analytic functions and its corollaries. □

Remarks.

1. Theorem 2 becomes a result on *all* power series after the change of variable $z \to z - z_0$.
2. According to Theorem 2, deriving a PS cannot *decrease* its radius of convergence. Could it possibly increase?
3. We know that analytic functions on disks have analytic antiderivatives. Might we be able to integrate term-by-term as well?

Hadamard's Formula

It turns out that there is a simple formula for the radius of convergence that will answer these questions.

Theorem 3 (Hadamard's Formula)

The radius of convergence of the power series $\sum a_k z^k$ *is given by*

$$\left(\limsup_{k\to\infty} \sqrt[k]{|a_k|}\right)^{-1}.$$

Proof. Let $L = \limsup \sqrt[k]{|a_k|} < \infty$.

Suppose that $0 < |z| < 1/L$.

Choose $L < R < 1/|z|$. Note that $|Rz| < 1$.

Then there is a K so that $\sup_{k \geq K} \sqrt[k]{|a_k|} < R$.

In particular, $\sqrt[k]{|a_k|} < R$ for $k \geq K$. Hence

$$\left|a_k z^k\right| < |Rz|^k \quad \text{for} \quad k \geq K,$$

and $\sum \left|a_k z^k\right|$ is eventually dominated by the convergent geometric series $\sum |Rz|^k$.

Thus, $\sum a_k z^k$ converges when $|z| < 1/L$.

On the other hand, suppose $|z| > 1/L$, so that $1/|z| < L$.

We know that $L \leq \sup_{k \geq K} \sqrt[k]{|a_k|}$ for all K, by definition.

So for each K we can choose $n_K \geq K$ so that $1/|z| < \sqrt[n_K]{|a_{n_K}|}$.

This means we can find arbitrarily large values of k for which $1/|z| < \sqrt[k]{|a_k|}$.

That is, $|a_k z^k| > 1$ infinitely often, so that $a_k z^k \not\to 0$, and $\sum a_k z^k$ diverges.

□

Examples

Example 4

Show that $\sum a_k z^k$ and its derived series $\sum k a_k z^{k-1}$ have the same radius of convergence.

Solution. We begin by writing

$$\sum_{k=0}^{\infty} k a_k z^{k-1} = \frac{1}{z} \sum_{k=0}^{\infty} k a_k z^k,$$

and note that both series converge for the same values of z.
We apply Hadamard in the second:

$$\limsup_{k\to\infty} \sqrt[k]{|k a_k|} = \limsup_{k\to\infty} \sqrt[k]{|a_k|}$$

because $\lim \sqrt[k]{k} = 1$.

It follows that the original series and the derived series have the same value of R (the reciprocal of the common lim sup). □

Example 5

Show that $\sum a_k z^k$ and the integrated series $\sum \frac{a_k}{k+1} z^{k+1}$ have the same radius of convergence.

Solution. The solution is nearly identical, except that we use the fact that

$$\limsup_{k\to\infty} \sqrt[k]{|a_k/(k+1)|} = \limsup_{k\to\infty} \sqrt[k]{|a_k|},$$

since $\lim_{k\to\infty} \sqrt[k]{k+1} = 1$. □

Towers of Power Series

Suppose $f(z) = \sum a_k z^k$ has positive radius of convergence R.

Inductively applying Example 4 in Theorem 2, we find that $f(z)$ is analytic for $|z| < R$, all of its derivatives are given by formal differentiation of PS, and all have radius of convergence R as well.

By Example 5, $F(z) = \sum \frac{a_k}{k+1} z^{k+1}$ is also analytic, with radius R. By Theorem 2, $F'(z) = f(z)$.

Again by induction, we can formally antidifferentiate $f(z)$ arbtrarily often to obtain repeated PS antiderivatives of $f(z)$, all with radius R.

This yields a tower of power series, all with common radius of convergence R:

$$\cdots \xrightarrow[d/dx]{} f^{(-2)} \xrightarrow[d/dx]{} f^{(-1)} \xrightarrow[d/dx]{} f \xrightarrow[d/dx]{} f' \xrightarrow[d/dx]{} f'' \xrightarrow[d/dx]{} \cdots$$

Each term is obtained by formally differentiating or integrating the adjacent terms.

Example 6

Find a power series representation for $\operatorname{Arctan} z$ centered at $z_0 = 0$.

Solution. We know that

$$\frac{d}{dz} \operatorname{Arctan} z = \frac{1}{1+z^2} = \sum_{k=0}^{\infty} (-1)^k z^{2k}$$

for $|z^2| < 1 \iff |z| < 1$.

It follows that

$$\operatorname{Arctan} z = \int \sum_{k=0}^{\infty} (-1)^k z^{2k}\, dz = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1} + C,$$

for $|z| < 1$. Setting $z = 0$ we find that $C = 0$, so that

$$\boxed{\operatorname{Arctan} z = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1} \quad \text{for} \quad |z| < 1}.$$