Consequences of the Local Version of Cauchy's Integral Formula

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Complex Variables

Recall

Last time we proved:

Theorem 1 (Local Cauchy Integral Formula)

Suppose $f(z)$ is analytic on an open disk D. If γ is a simple loop in D and z_0 is inside γ, then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

We can use the local Cauchy integral formula to obtain some very interesting *global* results.

Differentiation

Let $\Omega \subset \mathbb{C}$ be a domain and suppose $f : \Omega \to \mathbb{C}$ is analytic.

Let $z_0 \in \Omega$ and choose $r_0 > 0$ so that the disk $D = \{|z - z_0| < r_0\}$ is contained in Ω.

Fix an r with $0 < r < r_0$. The circle $C_r = \{|z - z_0| = r\}$ is contained in D, and Cauchy's integral formula tells us

$$f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w}\, dz \quad \text{for} \quad |w - z_0| < r.$$

Assuming we can differentiate under the integral sign, we find that

$$f'(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - w)^2}\, dz \quad \text{for} \quad |w - z_0| < r.$$

Can we justify this operation?

Fundamental Lemma

Let γ be a piecewise C^1 path in $\mathbb{C}$ and suppose $g(z)$ is continuous on γ. Let $n \in \mathbb{N}$.

For $w \in \mathbb{C} \setminus \gamma$ define

$$F(w) = \int_\gamma \frac{g(z)}{(z-w)^n}\, dz.$$

Lemma 1

The function $F(w)$ is analytic on $\mathbb{C} \setminus \gamma$, with

$$F'(w) = \frac{d}{dw}\int_\gamma \frac{g(z)}{(z-w)^n}\, dz = \int_\gamma \frac{\partial}{\partial w}\frac{g(z)}{(z-w)^n}\, dz$$
$$= n\int_\gamma \frac{g(z)}{(z-w)^{n+1}}\, dz.$$

Applied to

$$f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} \, dz \quad \text{for} \quad |w - z_0| < r,$$

Lemma 1 immediately yields

$$f'(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - w)^2} \, dz \quad \text{for} \quad |w - z_0| < r. \tag{1}$$

But Lemma 1 also applies to the RHS of (1). Thus f' is analytic on $|w - z_0| < r$, and

$$f''(w) = \frac{2}{2\pi i} \int_{C_r} \frac{f(z)}{(z - w)^3} \, dz \quad \text{for} \quad |w - z_0| < r.$$

Analytic Implies C^1

Since $z_0 \in \Omega$ was arbitrary, we come to the following conclusion.

Theorem 2

Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be analytic. Then f' is analytic on Ω.

We can now deduce a result we've been alluding to all semester.

Corollary 1

If f is analytic on a domain Ω, then $f \in C^1(\Omega)$.

Proof. Since f' is analytic on Ω, it is continuous. □

Remark. This means that the C^1 version of Cauchy's theorem is valid for all analytic functions!

Analytic Implies C^∞

Theorem 2 tells us that if f is analytic on Ω, then so is f'.

We may therefore apply Theorem 2 to f' to conclude that f'' is analytic on Ω.

We can continue in this manner indefinitely, arriving at the following conclusion.

Corollary 2

If f is analytic on a domain Ω, then f is infinitely differentiable on Ω.

Remark. Strictly speaking, this should be proven using Theorem 2 and induction.

Proof of Lemma 1

Let's return to the proof of Lemma 1.

We assume $n = 1$ for convenience.

The general case is similar, but is more algebraically involved.

Fix $z_0 \in \mathbb{C} \setminus \gamma$.

To compute $F'(z_0)$ we first we look at the difference quotient:

$$\frac{F(w) - F(z_0)}{w - z_0} = \frac{1}{w - z_0} \int_\gamma g(z) \left(\frac{1}{z - w} - \frac{1}{z - z_0} \right) dz$$

$$= \int_\gamma \frac{g(z)}{(z - w)(z - z_0)} \, dz.$$

We now compare the difference quotient to the purported derivative:

$$\frac{F(w)-F(z_0)}{w-z_0} - \int_\gamma \frac{g(z)}{(z-z_0)^2}\,dz$$
$$= \int_\gamma \frac{g(z)}{(z-w)(z-z_0)}\,dz - \int_\gamma \frac{g(z)}{(z-z_0)^2}\,dz$$
$$= \int_\gamma g(z)\left(\frac{1}{(z-w)(z-z_0)} - \frac{1}{(z-z_0)^2}\right)\,dz$$
$$= (w-z_0)\int_\gamma \frac{g(z)}{(z-w)(z-z_0)^2}\,dz.$$

Let $R = \min\{|z - z_0| : z \in \gamma\} > 0$. Then $|z - z_0| \geq R$ for all $z \in \gamma$.

If $|z_0 - w| < R/2$, then for any $z \in \gamma$

$$|z - w| = |z - z_0 + z_0 - w| \geq |z - z_0| - |z_0 - w| \geq R - \frac{R}{2} = \frac{R}{2}.$$

We therefore have

$$\left|\frac{1}{(z - w)(z - z_0)^2}\right| \leq \frac{1}{(R/2)R^2} = \frac{2}{R^3}.$$

Let M denote the maximum value of $g(z)$ on γ, and $L(\gamma)$ the arc length of γ.

Putting everything together we find that

$$\left|\frac{F(w)-F(z_0)}{w-z_0}-\int_\gamma \frac{g(z)}{(z-z_0)^2}\,dz\right|$$
$$=|w-z_0|\cdot\left|\int_\gamma \frac{g(z)}{(z-w)(z-z_0)^2}\,dz\right|$$
$$\leq |w-z_0|\frac{2M\,L(\gamma)}{R^3}$$

for $|w-z_0|<R/2$.

As $w\to z_0$, the RHS can be made arbitrarily small. Thus

$$F'(z_0)=\lim_{w\to z_0}\frac{F(w)-F(z_0)}{w-z_0}=\int_\gamma \frac{g(z)}{(z-z_0)^2}\,dz.$$

□

Cauchy's Estimates

Returning to the local situation, recall that if f is analytic on $|z - z_0| < r_0$ and $0 < r < r_0$, then

$$f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w}\, dz \quad \text{for} \quad |z_0 - w| < r,$$

where $C_r = \{|z - z_0| = r\}$.

Repeated (inductive) application of Lemma 1 (differentiation under the integral) leads to the formulae

$$f^{(k)}(w) = \frac{k!}{2\pi i} \int_{C_r} \frac{f(z)}{(z - w)^{k+1}}\, dz \quad \text{for} \quad |z_0 - w| < r,$$

for $k \in \mathbb{N}_0$.

Suppose that $|f(z)| \leq M$ for $|z - z_0| = r$. Then for any $k \in \mathbb{N}_0$,

$$\left|f^{(k)}(z_0)\right| = \left|\frac{k!}{2\pi i}\int_{C_r}\frac{f(z)}{(z-z_0)^{k+1}}\,dz\right| \leq \frac{k!}{2\pi}\frac{M}{r^{k+1}}2\pi r = \frac{k!}{r^k}M.$$

These are *Cauchy's estimates*.

Theorem 3 (Cauchy's Estimates)

Suppose f is analytic on $|z - z_0| < r_0$ and $0 < r < r_0$. If $|f(z)| \leq M$ for $|z - z_0| = r$, then

$$\left|f^{(k)}(z_0)\right| \leq \frac{k!}{r^k}M,$$

for all $k \in \mathbb{N}_0$.

Liouville's Theorem

Now suppose f is entire and that $|f(z)| \leq M_0$ for all $z \in \mathbb{C}$ (f is *bounded*).

For any $r > 0$ and $z_0 \in \mathbb{C}$, f is analytic on $|z - z_0| < r + 1$.

We may therefore take $r_0 = r + 1$ and apply Cauchy's estimates with $M = M_0$.

When $k = 1$, this yields

$$\left|f'(z_0)\right| \leq \frac{M_0}{r}$$

for all $r > 0$.

Letting $r \to \infty$, we conclude that $f'(z_0) = 0$.

Since $z_0 \in \mathbb{C}$ was arbitrary, this means $f' \equiv 0$.

Hence, f is constant! We have proven the following result.

Theorem 4 (Liouville's Theorem)

A bounded entire function is constant.

Remark. Theorem 4 is actually due to Cauchy as well, and was misattributed to Liouville by Borchardt.

Example 1

Suppose f is entire and $\operatorname{Re} f(z) \leq M$ for all $z \in \mathbb{C}$. Show that f is constant.

Solution. Let $g(z) = e^{f(z)}$.

Then g is also entire and $|g(z)| = e^{\operatorname{Re} f(z)} \leq e^M$.

By Liouville's theorem, $g \equiv C$ for some $C \in \mathbb{C}$.

So the image of f must be among the (discrete) values of $\log C$, which are given by $w_n = \operatorname{Log} C + 2n\pi i$, $n \in \mathbb{Z}$.

Since f is continuous and $\mathbb{C}$ is connected, $f(\mathbb{C})$ is also connected.

It follows that $f(\mathbb{C}) = \{w_m\}$ for some $m \in \mathbb{Z}$. That is, f is constant. □

Cauchy's Integral Formula (Again)

Finally, we prove the (global) Cauchy integral formula.

Theorem 5 (Cauchy's Integral Formula)

Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be analytic. If γ is a simple loop in Ω and z_0 is inside γ, then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, dz.$$

Proof. Choose $r_0 > 0$ so that the disk $|z - z_0| < r_0$ is inside γ.

Then choose r so that $0 < r < r_0$ and let C_r denote the circle $|z - z_0| = r$.

According to the local Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_0}\, dz.$$

Let Ω' denote the region between C_r and γ. Then $\partial\Omega' = \gamma - C_r$.

Because $f(z)/(z - z_0)$ is analytic on Ω', Cauchy's theorem implies

$$\begin{aligned} 0 &= \int_{\partial\Omega'} \frac{f(z)}{z - z_0}\, dz = \int_{\gamma} \frac{f(z)}{z - z_0}\, dz - \int_{C_r} \frac{f(z)}{z - z_0}\, dz \\ &= \int_{\gamma} \frac{f(z)}{z - z_0}\, dz - 2\pi i\, f(z_0), \end{aligned}$$

and the integral formula follows. □

Cauchy's Integral Formula for Derivatives

Application of Lemma 1 to Cauchy's integral formula yields:

Corollary 3

With Ω, γ and z_0 as above, for any $k \in \mathbb{N}$,

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{k+1}}\, dz.$$

Cauchy's integral formula can be used "in reverse" to evaluate integrals.

Example 2

Evaluate the integral $\displaystyle\int_\gamma \frac{\cos z}{z^3}\, dz$, where γ is any simple loop enclosing $z = 0$.

Solution. We apply Cauchy's integral formula for derivatives with $f(z) = \cos z$, $z_0 = 0$ and $k = 2$.

This yields

$$\frac{d^2}{dz^2}\cos z\bigg|_{z=0} = \frac{2}{2\pi i}\int_\gamma \frac{\cos z}{z^3}\,dz.$$

Since $\frac{d^2}{dz^2}\cos z = -\cos z$ and $\cos 0 = 1$, we conclude that

$$\int_\gamma \frac{\cos z}{z^3}\,dz = -i\pi.$$

□