Cauchy's Integral Formula

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Complex Variables

Recall

We've have proven:

Theorem 1 (Cauchy's Theorem for a Disk)

Let $z_0 \in \mathbb{C}$ and $r > 0$. Suppose $f(z)$ is analytic on the disk $D = \{z : |z - z_0| < r\}$. Then:

1. *f has an antiderivative in D;*
2. $\displaystyle\int_\gamma f(z)\,dz = 0$ *for any loop γ in D.*

Essential to the proof was the following result.

Theorem 2 (Cauchy's Theorem for Rectangles)

Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be analytic. If R is a closed rectangular region in Ω, then $\int_{\partial R} f(z)\,dz = 0$.

Strengthened Cauchy's Theorem for a Disk

For our purposes, we will require the following slightly stronger version of Cauchy's theorem for a disk.

Theorem 3 (Strong Cauchy's Theorem for a Disk)

Let $z_0 \in \mathbb{C}$ and $r > 0$. Suppose $f(z)$ is continuous on the disk $D = \{z : |z - z_0| < r\}$ and analytic on $D \setminus \{z_1\}$, for some $z_1 \in D$. Then:

1. *f has an antiderivative in D;*
2. *$\displaystyle\int_\gamma f(z)\,dz = 0$ for any loop γ in D.*

The strong version of Cauchy's theorem follows from an appropriate strengthening of Cauchy's theorem for rectangles.

Theorem 4 (Strong Cauchy's Theorem for Rectangles)

Let $\Omega \subset \mathbb{C}$ be a domain and let $z_0 \in \Omega$. Suppose $f : \Omega \to \mathbb{C}$ is continuous on Ω and analytic on $\Omega \setminus \{z_0\}$. If R is a closed rectangular region in Ω, then $\int_{\partial R} f(z)\, dz = 0$.

Theorem 4 is an immediate consequence of two lemmas.

Lemma 1

Let $\Omega \subset \mathbb{C}$ be a domain and let $z_0 \in \Omega$. Suppose $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is analytic and satisfies $\lim_{z \to z_0}(z - z_0)f(z) = 0$. If R is any closed rectangular region in Ω and $z_0 \notin \partial R$, then $\int_{\partial R} f(z)\, dz = 0$.

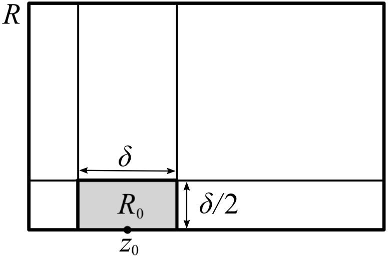
Remark. The condition $\lim_{z \to z_0}(z - z_0)f(z) = 0$ is satisfied, e.g., if f is continuous at z_0.

Lemma 2

Let $\Omega \subset \mathbb{C}$ be a domain and let $z_0 \in \Omega$. Suppose $f : \Omega \to \mathbb{C}$ is continuous on Ω and analytic on $\Omega \setminus \{z_0\}$. If R is a closed rectangular region in Ω and $z_0 \in \partial R$, then $\int_{\partial R} f(z)\, dz = 0$.

Proof of Lemma 1: HW. □

Proof of Lemma 2: Subdivide R into subrectangles as shown:



Then

$$\int_{\partial R} f(z)\,dz = \sum_{R'} \int_{\partial R'} f(z)\,dz, \tag{1}$$

where R' runs over all of the subrectangles.

If R' is a white subrectangle, then f is analytic on R', and the original rectangular Cauchy's theorem implies $\int_{\partial R'} f(z)\,dz = 0$.

Thus (1) becomes

$$\int_{\partial R} f(z)\,dz = \int_{\partial R_0} f(z)\,dz.$$

We will show that we can make the RHS arbitrarily small by choosing δ appropriately.

Use continuity to choose $\delta_0 > 0$ so that $|f(z) - f(z_0)| < 1$ whenever $|z - z_0| < \delta_0$.

Let $0 < \delta < \delta_0$.

Then for $|z - z_0| < \delta$ we have

$$|f(z)| = |f(z) - f(z_0) + f(z_0)| \leq 1 + |f(z_0)|.$$

Since ∂R_0 is contained in $|z - z_0| < \delta$, an ML estimate then gives

$$\left|\int_{\partial R} f(z)\,dz\right| = \left|\int_{\partial R_0} f(z)\,dz\right| \leq 3\delta(1 + |f(z_0)|).$$

As $0 < \delta < \delta_0$ was arbitrary, this implies that $\int_{\partial R} f(z)\,dz = 0$. □

As noted, the strong Cauchy theorem for rectangles follows at once from Lemmas 1 and 2 (the exceptional point z_0 either lies on ∂R or it doesn't).

The strong Cauchy theorem for a disk follows by substituting the strong Cauchy theorem for rectangles in the proof of the "weak" Cauchy theorem for a disk.

We will eventually use the strong Cauchy theorem on a disk to prove the *Cauchy integral formula.*

Winding Numbers

Definition

Let $z_0 \in \mathbb{C}$ and suppose γ is any loop that avoids z_0. The *index* (or *winding number*) of γ with respect to z_0 is

$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

Lemma 3

Let $z_0 \in \mathbb{C}$ and let γ be a loop avoiding z_0. Then $I(\gamma, z_0) \in \mathbb{Z}$.

Proof. Suppose $\gamma : [a, b] \to \mathbb{C}$ is piecewise C^1, so that

$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(\tau)}{\gamma(\tau) - z_0}\, d\tau.$$

Let

$$g(t) = \int_a^t \frac{\gamma'(\tau)}{\gamma(\tau) - z_0}\, d\tau.$$

FTOC implies

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0}$$

on the intervals where $\gamma'(t)$ is continuous. We then we have

$$\begin{aligned}
\frac{d}{dt} e^{-g(t)}(\gamma(t) - z_0) &= -g'(t)e^{-g(t)}(\gamma(t) - z_0) + e^{-g(t)}\gamma'(t) \\
&= e^{-g(t)}\left(-g'(t)(\gamma(t) - z_0) + \gamma'(t)\right) \\
&= e^{-g(t)}\left(-\gamma'(t) + \gamma'(t)\right) \\
&= 0,
\end{aligned}$$

which means $e^{-g(t)}(\gamma(t) - z_0)$ is *piecewise* constant.

But $e^{-g(t)}(\gamma(t) - z_0)$ is continuous, so $e^{-g(t)}(\gamma(t) - z_0) = C$.

Since $g(a) = 0$ we find that $C = e^0(\gamma(a) - z_0) = \gamma(a) - z_0$. Thus

$$e^{-g(t)}(\gamma(t) - z_0) = \gamma(a) - z_0. \tag{2}$$

Since $\gamma(b) = \gamma(a)$ and $g(b) = 2\pi i \cdot I(\gamma; z_0)$, setting $t = b$ in (2) yields

$$e^{-2\pi i \cdot I(\gamma;z_0)}(\gamma(a) - z_0) = \gamma(a) - z_0 \;\Rightarrow\; e^{2\pi i \cdot I(\gamma;z_0)} = 1,$$

since $\gamma(a) \neq z_0$. This implies that

$$\begin{aligned} 2\pi i \cdot I(\gamma; z_0) \equiv 0 \pmod{2\pi i} &\iff I(\gamma; z_0) \equiv 0 \pmod{1} \\ &\iff I(\gamma; z_0) \in \mathbb{Z}. \end{aligned}$$

□

The index $I(\gamma; z_0)$ measures how many times γ "wraps around" z_0.

Example 1

Show that if γ is a positively oriented simple loop avoiding z_0, then

$$I(\gamma; z_0) = \begin{cases} 1, & \text{if } z_0 \text{ is inside } \gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Solution. Since $\frac{d}{dz}\frac{1}{z-z_0} = \frac{-1}{(z-z_0)^2}$ is continuous away from z_0, we may apply the C^1 version of Cauchy's theorem.

If z_0 is outside γ, then $\frac{1}{z-z_0}$ is analytic on and inside γ. Thus

$$I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = 0,$$

by Cauchy's theorem.

If z_0 is inside γ, we can choose an $r > 0$ so that $C_r = \{|z - z_0| = r\}$ is also inside γ.

$\frac{1}{z-z_0}$ is analytic between γ and $|z - z_0| = r$, so we may apply Cauchy's theorem:

(picture)

This yields

$$0 = \int_\gamma \frac{dz}{z - z_0} + \int_{-C_r} \frac{dz}{z - z_0} = \int_\gamma \frac{dz}{z - z_0} - 2\pi i,$$

by an earlier example. The result follows. □