Cauchy's Theorem

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Complex Variables

Recall

Last time we used Green's theorem to prove the following result.

Theorem 1 (Cauchy's Theorem)

Let Ω be a bounded domain with piecewise smooth boundary $\partial\Omega$. If $f : \Omega \to \mathbb{C}$ is analytic, C^1, and f extends smoothly to $\partial\Omega$, then

$$\int_{\partial\Omega} f(z)\,dz = 0.$$

We also discussed the need to remove the C^1 hypothesis.

This can be done at the expense of limiting Ω to being a rectangle.

The "rectangular" Cauchy's theorem can then be used to prove more general, C^1-free versions.

Preliminary Result

Lemma 1

Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$. If f is (complex) differentiable at $z_0 \in \Omega$, then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z), \tag{1}$$

where

$$\lim_{z \to z_0} \frac{E(z)}{z - z_0} = 0.$$

Proof. Let $E(z) = f(z) - f(z_0) - f'(z_0)(z - z_0)$. Then (1) holds and

$$\lim_{z \to z_0} \frac{E(z)}{z - z_0} = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right) = f'(z_0) - f'(z_0) = 0.$$

□

Cauchy's Theorem Revisited

Theorem 2 (Cauchy's Theorem for Rectangles)

Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be analytic. If R is a closed rectangular region in Ω, then

$$\int_{\partial R} f(z)\, dz = 0.$$

Proof. Quarter R into 4 identical subrectangles, $R_j^{(1)}$, $j = 1, 2, 3, 4$. If we orient all contours positively, we then have

$$\int_{\partial R} f(z)\, dz = \sum_{j=1}^{4} \int_{\partial R_j^{(1)}} f(z)\, dz,$$

because the integrals along the portions of the $\partial R_j^{(1)}$ interior to R cancel out.

Thus there is a j so that

$$\frac{1}{4}\left|\int_{\partial R} f(z)\,dz\right| \leq \left|\int_{\partial R_j^{(1)}} f(z)\,dz\right|,$$

or equivalently, setting $R^{(1)} = R_j^{(1)}$,

$$\left|\int_{\partial R} f(z)\,dz\right| \leq 4\left|\int_{\partial R^{(1)}} f(z)\,dz\right|$$

Now recursively continue this procedure.

Given rectangle $R^{(k)}$, subdivide it into fourths $R_j^{(k+1)}$ as above, and note that

$$\int_{\partial R^{(k)}} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial R_j^{(k+1)}} f(z)\,dz.$$

Then choose j so that $R^{(k+1)} := R_j^{(k+1)}$ satisfies

$$\left|\int_{\partial R^{(k)}} f(z)\,dz\right| \le 4\left|\int_{\partial R^{(k+1)}} f(z)\,dz\right|. \qquad (2)$$

The result is a nested sequence

$$R = R^{(0)} \supset R^{(1)} \supset R^{(2)} \supset \cdots$$

of rectangles, each with dimensions half the size of the preceding rectangle, and satisfying (2) for $k \ge 0$.

Stringing the inequalities (2) together yields

$$\left|\int_{\partial R} f(z)\,dz\right| \le 4^k\left|\int_{\partial R^{(k)}} f(z)\,dz\right|,$$

for $k \ge 0$.

Let $a \times b$ be the dimensions of R. Then $R^{(k)}$ is $\frac{a}{2^k} \times \frac{b}{2^k}$.

Because R is compact, and the $R^{(k)}$ are closed, nested and nonempty, their intersection is nonempty.

The intersection cannot contain two points because the dimensions of the $R^{(k)}$ become arbitrarily small.

Thus $\displaystyle\bigcap_{k=0}^{\infty} R^{(k)} = \{z_0\}$ for some $z_0 \in R$.

Use Lemma 1 to write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z),$$

where

$$\lim_{z \to z_0} \frac{E(z)}{z - z_0} = 0.$$

Then

$$\begin{aligned}\int_{\partial R^{(k)}} f(z)\,dz &= \int_{\partial R^{(k)}} f(z_0) + f'(z_0)(z - z_0)\,dz + \int_{\partial R^{(k)}} E(z)\,dz\\ &= \int_{\partial R^{(k)}} E(z)\,dz\end{aligned}$$

by the fundamental theorem of calculus, since $f(z_0) + f'(z_0)(z - z_0)$ has an antiderivative and $\partial R^{(k)}$ is a closed path.

Let $\epsilon > 0$ and choose $\delta > 0$ so that $|E(z)/(z - z_0)| < \epsilon$ for $0 < |z - z_0| < \delta$.

Choose K so large that $R^{(k)}$ is contained in $|z - z_0| < \delta$ for $k \geq K$.

Then for $k \geq K$ we have

$$\left|\int_{\partial R^{(k)}} f(z)\,dz\right| = \left|\int_{\partial R^{(k)}} E(z)\right| \leq \int_{\partial R^{(k)}} |E(z)|\,|dz| \leq \int_{\partial R^{(k)}} \epsilon|z - z_0|\,|dz| \leq \epsilon \Delta_k P_k,$$

where $\Delta_k = \frac{\sqrt{a^2+b^2}}{2^k}$ is the length of the diagonal of $R^{(k)}$ and $P_k = \frac{a+b}{2^{k-1}}$ is the perimeter of $R^{(k)}$.

Finally, for $k \geq K$, we have

$$\left|\int_{\partial R} f(z)\,dz\right| \leq 4^k \left|\int_{R^{(k)}} f(z)\,dz\right|$$
$$\leq 4^k \epsilon \frac{\sqrt{a^2+b^2}}{2^k} \frac{(a+b)}{2^{k-1}}$$
$$= 2\epsilon(a+b)\sqrt{a^2+b^2}.$$

As this is true for any $\epsilon > 0$, we conclude that

$$\int_{\partial R} f(z)\,dz = 0.$$

□

Cauchy's Theorem for a Disk

We can now generalize Cauchy's theorem to arbitrary closed paths, provided we assume Ω is a disk. For our applications this will be sufficient.

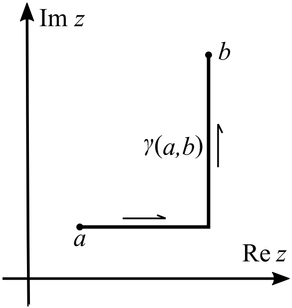
Theorem 3 (Cauchy's Theorem for a Disk)

Let $z_0 \in \mathbb{C}$ and $r > 0$. Suppose $f(z)$ is analytic on the disk $D = \{z : |z - z_0| < r\}$. Then:

1. *f has an antiderivative in D;*
2. $\displaystyle\int_\gamma f(z)\,dz = 0$ *for any loop γ in D.*

Proof. It suffices to prove **1**, since **2** follows from **1** and FTOC.

Given $a, b \in \mathbb{C}$, let $\gamma(a, b)$ denote the L-shaped path from a to b, consisting of a horizontal line segment followed by a vertical segment:



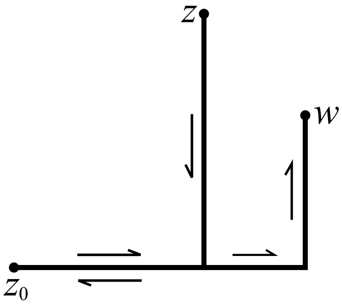
For $z \in D$, the path $\gamma(z_0, z)$ lies in D, and we define

$$F(z) = \int_{\gamma(z_0,z)} f(\zeta)\, d\zeta.$$

Notice that for $w \in D$,

$$F(w) - F(z) = \int_{\gamma(z,w)} f(\zeta)\, d\zeta + \int_{\partial R} f(\zeta)\, d\zeta$$

for a certain rectangular region R in D (there are several cases to check):



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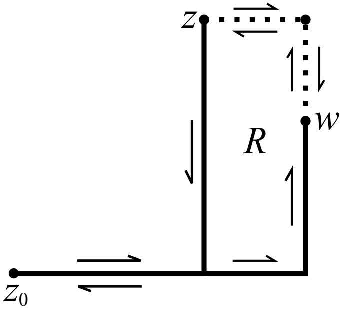

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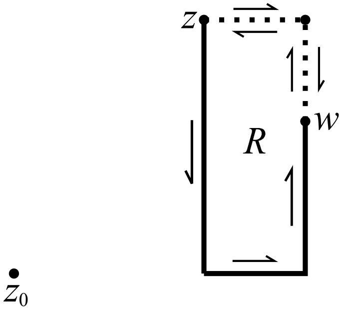
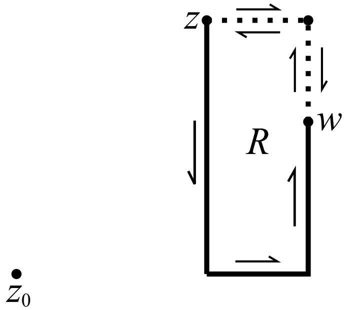


By Cauchy's Theorem for Rectangles, $\int_{\partial R} f(\zeta)\,d\zeta = 0$. Thus

$$F(w) - F(z) = \int_{\gamma(z,w)} f(\zeta)\,d\zeta.$$

Graphically:



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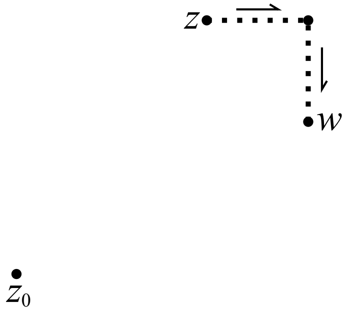

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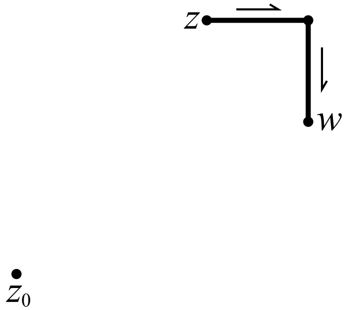

Therefore

$$\begin{aligned}\left|\frac{F(w)-F(z)}{w-z}-f(z)\right| &= \left|\frac{1}{w-z}\int_{\gamma(z,w)} f(\zeta)\,d\zeta - f(z)\right| \\ &= \frac{1}{|w-z|}\left|\int_{\gamma(z,w)} f(\zeta)\,d\zeta - f(z)(w-z)\right| \\ &= \frac{1}{|w-z|}\left|\int_{\gamma(z,w)} f(\zeta)\,d\zeta - f(z)\int_{\gamma(z,w)} d\zeta\right| \\ &= \frac{1}{|w-z|}\left|\int_{\gamma(z,w)} f(\zeta)-f(z)\,d\zeta\right|.\end{aligned}$$

Let $\epsilon > 0$ and use the continuity of f at z to choose $\delta > 0$ so that $|f(\zeta) - f(z)| < \epsilon$ for $0 < |\zeta - z| < \delta$.

Then for $0 < |w - z| < \delta$ (in D) we have

$$\frac{1}{|w-z|}\left|\int_{\gamma(z,w)} f(\zeta) - f(z)\, d\zeta\right| \leq \frac{\epsilon \cdot \ell(\gamma(z,w))}{|w-z|} \leq \epsilon\sqrt{2},$$

since $\ell(\gamma(z,w)) \leq \sqrt{2}|z-w|$ by the Cauchy-Schwartz inequality.

Since $\epsilon > 0$ is arbitrary, this proves that

$$F'(z) = \lim_{w\to z} \frac{F(w) - F(z)}{w - z} = f(z).$$

□