Conformal Maps

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Complex Variables

Parametric Curves

Definition

A *(parametric) curve* in $\mathbb{C}$ is a continuous function $\gamma : (0,1) \to \mathbb{C}$. If $x(t) = \operatorname{Re} \gamma(t)$ and $y(t) = \operatorname{Im} \gamma(t)$, we say γ is a C^k *curve* provided $x(t), y(t) \in C^k((0,1))$.

Remarks.

1. One frequently identifies γ with it image $C = \gamma((0,1))$. Strictly speaking, however, the former is a *description* of the latter.
2. Because all open intervals in $\mathbb{R}$ are diffeomorphic, $(0,1)$ can be replaced by any other open interval.

Examples

1. $\gamma(t) = t + i(mt + b)$, $t \in \mathbb{R}$, is a smooth parametrization of the line $y = mx + b$.
2. $\gamma(t) = te^{i\theta_0}$, $t \in (0, \infty)$, is a smooth parametrization of the ray $\arg z = \theta_0$.
3. $\gamma(t) = e^{it}$, $t \in \mathbb{R}$, is a smooth parametrization of the unit circle.
4. $\gamma(t) = t^2 + it$, $t \in \mathbb{R}$, parametrizes the rightward opening parabola $x = y^2$.

Tangent Vectors

Definition

If $\gamma(t) = x(t) + iy(t)$ is a C^1 curve, its *derivative* is

$$\gamma'(t) = x'(t) + iy'(t),$$

which is the familiar *tangent vector* in $\mathbb{R}^2$.

Notice that

$$\begin{aligned}\gamma'(t) &= \left(\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}\right) + i\left(\lim_{h\to 0} \frac{y(t+h) - y(t)}{h}\right)\\ &= \lim_{h\to 0} \frac{x(t+h) + iy(t+h) - x(t) - iy(t)}{h}\\ &= \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.\end{aligned}$$

Compositions

Let $\Omega \subset \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be analytic.

If γ is a curve in Ω, then its image under f, namely $\Gamma = f \circ \gamma$, is also a curve.

If γ is C^1, then for any t:

$$\begin{aligned}
\Gamma'(t) &= \lim_{h\to 0} \frac{\Gamma(t+h) - \Gamma(t)}{h} \\
&= \lim_{h\to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h} \\
&= \lim_{h\to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{\gamma(t+h) - \gamma(t)} \cdot \frac{\gamma(t+h) - \gamma(t)}{h} \\
&= f'(\gamma(t))\gamma'(t),
\end{aligned}$$

so that Γ is also C^1 with derivative given by the chain rule.

Summary

Theorem 1 (Chain Rule for Curves)

Suppose f is analytic at z_0 and γ is any curve through z_0. Then the tangent vector to the image curve $\Gamma = f \circ \gamma$ at $f(z_0)$ is $f'(z_0)$ (complex) multiplied by the tangent vector to γ at z_0.

Geometrically speaking, this is saying that locally f dilates and rotates tangent vectors, by $|f'|$ and $\arg f'$ (resp.).

This provides a geometric interpretation of $f'(z_0)$.

This can also be seen in the Jacobian. If $f'(z_0) = u_x + iv_x = re^{i\theta}$:

$$J_f(z_0) = \begin{pmatrix} u_x & -v_x \\ v_x & v_y \end{pmatrix} = \underbrace{\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}}_{\text{scalar}} \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{rotation}}.$$

Angles

Now suppose curves γ_1 and γ_2 in Ω intersect at $z_0 = \gamma_1(t_1) = \gamma_2(t_2)$.

The angle φ between γ_1 and γ_2 at z_0 is defined to be the angle between the tangent vectors $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$, or

$$\varphi = \arg\left(\gamma_2'(t_2)\right) - \arg\left(\gamma_1'(t_1)\right) = \arg\left(\gamma_2'(t_2)/\gamma_1'(t_1)\right).$$

Notice that φ is not symmetric in γ_1 and γ_2. So our angles come with an *orientation*: counterclockwise from γ_1 to γ_2.

Let f be analytic at z_0. Let $\Gamma_j = f \circ \gamma_j$ be the image of γ_j under f ($j = 1, 2$). These intersect at $w_0 = f(z_0) = \Gamma_1(t_1) = \Gamma_2(t_2)$.

The tangent vectors at w_0 are $\Gamma'(t_j) = f'(z_0)\gamma_j'(t_j)$, by the chain rule.

If $f'(z_0) \neq 0$, the angle between the image curves Γ_1 and Γ_2 is

$$\begin{aligned} \varphi' &= \arg\left(\Gamma_2'(t_2)/\Gamma_1'(t_1)\right) \\ &= \arg\left(\frac{f'(z_0)\gamma_2'(t_2)}{f'(z_0)\gamma_1'(t_1)}\right) \\ &= \arg\left(\gamma_2'(t_2)/\gamma_1'(t_1)\right) \\ &= \varphi. \end{aligned}$$

Thus, f preserves the angles between tangent vectors at each point! (In both magnitude *and* orientation).

Conformality

Definition

Let $\Omega \subset \mathbb{R}^2$ be a domain. A map $\Omega \to \mathbb{R}^2$ is called *conformal* if it preserves angles (in magnitude and orientation) at every point in Ω.

We have now seen:

Theorem 2

An analytic function $f(z)$ is conformal everywhere that $f'(z) \neq 0$.

As an easy non-example we have:

Example 1

The function $f(z) = \overline{z}$ is *not* conformal.

This is simply because reflections *reverse* the orientations of angles.

Examples

Example 2

z^2 and $\sqrt{z}$

Example 3

e^z and $\log z$

Example 4

$\sin z$

Example 5

$1/z$

(See Maple)