Sequences and Series of Functions

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Complex Variables

Sequences of Functions

Let $E \subset \mathbb{C}$ and for each $n \in \mathbb{N}$ let $f_n : E \to \mathbb{C}$. There are two primary notions of the convergence of the sequence $\{f_n\}$.

Definition (Pointwise Convergence)

We say $\{f_n\}$ *converges (pointwise)* to $f : E \to \mathbb{C}$ if $f_n(z) \to f(z)$ for each $z \in E$.

Definition (Uniform Convergence)

We say $\{f_n\}$ *converges uniformly on* E to $f : E \to \mathbb{C}$ provided for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N$ and $z \in E$.

It is clear that uniform convergence on E implies pointwise convergence on E.

Examples

The mode of convergence of a sequence $\{f_n\}$ depends as much on f_n as it does on E.

Example 1

Let $f_n(z) = z^n$. Show that $f_n \to 0$ on $D = \{|z| < 1\}$, but not uniformly.

Solution. If $|z| < 1$, then $|f_n(z)| = |z^n| = |z|^n \to 0$ as $n \to \infty$.

Hence $f_n \to 0$ pointwise.

However, for any $N \in \mathbb{N}$, $z = 1/\sqrt[N]{2} \in D$ and $|f_N(z)| = 1/2$.

So $\{f_n\}$ cannot be made uniformly small on D. □

Example 2

Show that for any $0 < r < 1$, $f_n \to 0$ uniformly on $D_r = \{|z| \leq r\}$.

Solution. Let $\epsilon > 0$ and choose N so that $r^N < \epsilon$.

Then for $n \geq N$ and $z \in D_r$ one has

$$|f_n(z)| = |z|^n \leq r^n \leq r^N < \epsilon.$$

Thus $f_n \to 0$ uniformly on D_r. □

Cauchy Sequences

Let $E \subset \mathbb{C}$ and $f_n : E \to \mathbb{C}$ a sequence of functions.

Definition (Pointwise Cauchy)

We say $\{f_n\}$ is *pointwise Cauchy* if $\{f_n(z)\}$ is a Cauchy sequence for every $z \in E$.

Definition (Uniformly Cauchy)

We say $\{f_n\}$ is *uniformly Cauchy* on E if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $|f_m(z) - f_n(z)| < \epsilon$ for all $m, n \geq N$ and all $z \in E$.

Clearly uniformly Cauchy implies pointwise Cauchy, which is equivalent to pointwise convergence.

The notion of uniformly Cauchy will be useful when dealing with series of functions.

Uniformly Cauchy iff Uniformly Convergent

Let $E \subset \mathbb{C}$ and $f_n : E \to \mathbb{C}$ a sequence of functions.

Theorem 1

The sequence $\{f_n\}$ is uniformly Cauchy on E if and only if it is uniformly convergent on E.

Proof (sketch). ($\Leftarrow$) Exercise.

($\Rightarrow$) Suppose $\{f_n\}$ is uniformly Cauchy.

Then $\{f_n\}$ is pointwise Cauchy/convergent with limit function $f : E \to \mathbb{C}$.

We claim that $f_n \to f$ uniformly.

Let $\epsilon > 0$.

Choose $N \in \mathbb{N}$ so that $|f_m(z) - f_n(z)| < \epsilon$ for all $m, n \geq N$ and $z \in E$.

For each $z \in E$ choose $N_z \in \mathbb{N}$ so that $|f_n(z) - f(z)| < \epsilon$ for $n \geq N_z$.

Then for any $z \in E$ and $m = \max\{N, N_z\}$ one has

$$|f_N(z) - f(z)| \leq |f_N(z) - f_m(z)| + |f_m(z) - f(z)| < 2\epsilon.$$

Finally, for $n \geq N$ and $z \in E$ we have

$$|f_n(z) - f(z)| \leq |f_n(z) - f_N(z)| + |f_N(z) - f(z)| < 3\epsilon.$$

Since $\epsilon > 0$ was arbitrary, the result follows. □

Properties of Uniformly Convergent Sequences

Uniformly convergent sequences enjoy many properties that pointwise convergent sequences do not.

By a standard "3ϵ argument" as above one can prove that:

Theorem 2

If $f_n : E \to \mathbb{C}$ are continuous and converge uniformly to $f : E \to \mathbb{C}$, then f is continuous.

Of particular importance for us is the next property.

Theorem 3

Let γ be a piecewise C^1 path and suppose that $\{f_n\}$ are continuous and converge uniformly on γ. Then

$$\lim_{n\to\infty} \int_\gamma f_n(z)\, dz = \int_\gamma \lim_{n\to\infty} f_n(z)\, dz.$$

Proof

Theorem 2 implies that $f(z) = \lim_{n\to\infty} f_n(z)$ is continuous on γ.

So $\int_\gamma \lim_{n\to\infty} f_n(z)\,dz$ makes sense.

Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ so that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N$ and $z \in \gamma$.

Then for $n \geq N$ we have

$$\left|\int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz\right| = \left|\int_\gamma f_n(z) - f(z)\,dz\right| < \epsilon \cdot L(\gamma).$$

Since $L(\gamma)$ is fixed and $\epsilon > 0$ is arbitrary, we are finished. □

Series of Functions

Before moving on to analytic functions, we need an important result on uniform convergence of series.

Suppose $g_k : E \to \mathbb{C}$ is a sequence of functions and let

$$S_n(z) = \sum_{k=1}^{n} g_k(z)$$

denote the nth partial sum of $\sum g_k(z)$.

We say that the series $\sum g_k(z)$ *converges (pointwise or uniformly)* on E provided $\{S_n(z)\}$ does.

The M-Test

Perhaps the most important fundamental tool for studying the convergence of series of functions is:

Theorem 4 (Weierstrass M-test)

Let $g_k : E \to \mathbb{C}$ be a sequence of functions. Suppose there are constants $M_k \geq 0$ so that:

1. $|g_k(z)| \leq M_k$ *for all k and all $z \in E$;*
2. $\sum M_k$ *converges.*

Then $\sum g_k(z)$ converges absolutely and uniformly on E.

Proof. Absolute convergence at any $z \in E$ follows from the comparison test.

Proof

To prove uniform convergence, it suffices to prove that the sequence $\{S_n(z)\}$ of partial sums is uniformly Cauchy on E.

Let $\epsilon > 0$. Because $\sum M_k$ converges (absolutely), there is an $N \in \mathbb{N}$ so that

$$\sum_{k=m+1}^{n} M_k < \epsilon$$

for $n > m \geq N$. But then for any $z \in E$,

$$|S_m(z) - S_n(z)| = \left|\sum_{k=m+1}^{n} g_k(z)\right| \leq \sum_{k=m+1}^{n} M_k < \epsilon$$

for $n > m \geq N$. □

Examples

Example 3

Show that the geometric series $\sum z^k$ converges uniformly on $|z| \leq r$, for any $0 < r < 1$.

Solution. If $|z| \leq r$, then $|z^k| = |z|^k \leq r^k$.
Since $\sum r^k$ converges, the result follows from the M-test. □

Example 4

Show that the geometric series $\sum z^k$ does *not* converge uniformly on the open disk $|z| < 1$.

Solution. For $|z| < 1$ we have $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ and

$$\left| \frac{1}{1-z} - \sum_{k=0}^{n} z^k \right| = \left| \sum_{k=n+1}^{\infty} z^k \right| = \frac{|z|^{n+1}}{|1-z|}.$$

But the right-hand side cannot be made uniformly small for $|z| < 1$. (Exercise) □

In the context of series, we remark that Theorem 3 has the following restatement.

Theorem 5

Let γ be a piecewise C^1 path and suppose that $\{g_k\}$ are continuous on γ. If $\sum g_k$ converges uniformly on γ, then

$$\sum_{k=1}^{\infty} \int_{\gamma} g_k(z)\, dz = \int_{\gamma} \sum_{k=1}^{\infty} g_k(z)\, dz.$$

Normal Convergence

When dealing with sequences of *analytic* functions, there's another very useful type of convergence.

Definition

Let $\Omega \subset \mathbb{C}$ be a domain and let $f_n : \Omega \to \mathbb{C}$ be a sequence of functions. We say that $\{f_n\}$ *converges normally on* Ω provided $\{f_n\}$ converges uniformly on every compact subset of Ω.

Remarks.

1. Normal convergence on Ω is equivalent to uniform convergence on every *closed* subdisk of Ω.
2. Because $\{x\}$ is compact, $\{f_n\}$ converges pointwise to a common limit function f throughout Ω.

Example

Example 5

Show that the geometric series $\sum z^k$ converges normally on $|z| < 1$.

Solution. Let $E \subset \{|z| < 1\}$ be compact.

Then there is an $0 < r < 1$ so that $E \subset \{|z| \leq r\}$.

We have already seen that $\sum z^k$ converges uniformly on $\{|z| \leq r\}$, so it does the same on E. □

Properties of Normally Convergent Sequences

Theorem 6

Let $\Omega \subset \mathbb{C}$ be a domain and suppose $\{f_n\}$ is a sequence of analytic functions on Ω. If $\{f_n\}$ converges normally on Ω to f, then f is analytic on Ω.

Proof. We apply Morera's theorem.

Because each f_n is continuous, and $f_n \to f$ locally uniformly, f is continuous on Ω.

Let R be a closed rectangular region in Ω.

∂R is a compact loop, so Cauchy's theorem and Theorem 3 imply

$$0 = \lim_{n\to\infty} \int_{\partial R} f_n(z)\, dz = \int_{\partial R} \lim_{n\to\infty} f_n(z)\, dz = \int_{\partial R} f(z)\, dz.$$

It follows that f is analytic on Ω. □

If we try a little harder, we can actually say more. Because of the Cauchy integral formula, normal convergence "descends" to derivatives.

Theorem 7

Let $\Omega \subset \mathbb{C}$ and suppose $\{f_n\}$ is a sequence of analytic functions on Ω. If $\{f_n\}$ converges normally on Ω to f, then $\{f_n'\}$ converges normally on Ω to f'.

*Proof.*Let $D = \{|z - z_0| \leq R\}$ be a closed disk of radius R contained in Ω.

Because Ω is open, there is a $\rho > R$ so that the (slightly larger) open disk $|z - z_0| < \rho$ is also contained in Ω.

Fix $R < r < \rho$. Let $C_r = \{|z - z_0| = r\}$.

Because f and f_n are analytic on $|z - z_0| < \rho$, the Cauchy integral formula implies

$$f'(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta \quad \text{and} \quad f_n'(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f_n(\zeta)}{(\zeta - z)^2} \, d\zeta,$$

for $z \in D$ (which is inside C_r) and $n \in \mathbb{N}$.

Now on C_r we have

$$|\zeta - z| \geq |\zeta - z_0| - |z_0 - z| = r - |z - z_0| \geq r - R > 0.$$

Given $\epsilon > 0$, use normal convergence to find $N \in \mathbb{N}$ so that $|f_n(\zeta) - f(\zeta)| < \epsilon$ for all $\zeta \in C_r$ and all $n \geq N$.

The two Cauchy integral formulae above then imply that for $z \in D$ and $n \geq N$:

$$\begin{aligned} \left|f'(z) - f_n'(z)\right| &= \frac{1}{2\pi}\left|\int_{C_r} \frac{f(\zeta) - f_n(\zeta)}{(\zeta - z)^2}\, d\zeta\right| \\ &\leq \frac{1}{2\pi}\frac{\epsilon}{(r-R)^2}(2\pi r) \\ &= \frac{r\epsilon}{(r-R)^2}. \end{aligned}$$

Because R and r are fixed, the RHS can be made arbitrarily small, for *all* $z \in D$.

Hence $f_n' \to f'$ uniformly on D.

Since D was arbitrary, this proves that $f_n' \to f'$ normally on Ω. □

Normal Convergence of Higher Derivatives

Suppose $\{f_n\}$ are analytic and $f_n \to f$ normally on a domain Ω.

We have just seen that then $f_n' \to f'$ normally on Ω.

But f_n' is also sequence of analytic functions!

So we may apply Theorem 7 inductively to reach the following conclusion.

Corollary 1

Suppose $\{f_n\}$ is a sequence of analytic functions on a domain Ω. If $f_n \to f$ normally on Ω, then for all $m \in \mathbb{N}$ the sequence $\{f_n^{(m)}\}$ of mth derivatives converges normally on Ω to $f^{(m)}$.

Example

We have seen that $\sum z^k$ converges normally on $|z| < 1$, and that

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

By the corollary we then have

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty}\frac{d}{dz}z^k = \sum_{k=1}^{\infty} kz^{k-1},$$

normally for $|z| < 1$, which we derived earlier by other means.
In a similar manner we can show

$$\sum_{k=2}^{\infty} k(k-1)z^{k-2} = \frac{2}{(1-z)^3} \quad \text{for} \quad |z| < 1.$$