Harmonic Functions

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Complex Variables

Definition

The *Laplacian* (in two variables) is the differential operator

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Definition

Let $\Omega \subset \mathbb{R}^2$ be a domain and $u \in C^2(\Omega)$. We say u is *harmonic* provided

$$\Delta u = 0$$

throughout Ω.

Recall: If $u(x, y) \in C^2(\Omega)$, then

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

throughout Ω (*Clairaut's theorem*).

Example 1

The function $u(x,y) = x^3 - 3xy^2$ is harmonic on $\mathbb{C}$.

Indeed, u is clearly C^2 and

$$\left.\begin{array}{lcl} \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 & \Rightarrow & \dfrac{\partial^2 u}{\partial x^2} = 6x, \\ \dfrac{\partial u}{\partial y} = -6xy & \Rightarrow & \dfrac{\partial^2 u}{\partial y^2} = -6x, \end{array}\right\} \Rightarrow \Delta u = 0.$$

Example 2

The function $u(x,y) = \arctan(y/x)$ is harmonic for $x > 0$.

In this case

$$\frac{\partial u}{\partial x} = \frac{-y/x^2}{1 + (y/x)^2} = \frac{-y}{x^2 + y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}$$

and

$$\frac{\partial u}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}.$$

Thus $\Delta u = 0$ and u is harmonic.

More generally we have the following result.

Theorem 1

Let $\Omega \subset \mathbb{C}$ be a domain, $f : \Omega \to \mathbb{C}$, $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. If f is analytic and $u, v \in C^2(\Omega)$, then u and v are harmonic on Ω.

Remarks.

1. The C^2 hypothesis is actually unnecessary. As we will see, if f is analytic, then $\operatorname{Re} f$ and $\operatorname{Im} f$ are in fact C^∞.
2. We will see that *every* harmonic function is (locally) the real part of an analytic function.

Proof. By the C-R equations and Clairaut's theorem we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2},$$

which implies that $\Delta u = 0$. Similarly, $\Delta v = 0$. □

Definition

Let $\Omega \subset \mathbb{R}^2$ be a domain and let $u : \Omega \to \mathbb{R}$ be harmonic. We say that $v \in C^2(\Omega)$ is a *harmonic conjugate* of u (on Ω) provided u and v satisfy the C-R equations on Ω:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Equivalently, provided $f = u + iv$ is analytic on Ω.

Remarks.

1. If u is harmonic and v is a conjugate of u, then v is also harmonic.

2. Being "a harmonic conjugate of" is *not* symmetric. One *cannot* simply say that u and v are "harmonic conjugates of one another."

3. If v is a harmonic conjugate of u, then $-u$ is a harmonic conjugate of v: if $f = u + iv$ is analytic, then so is $-if = v - iu$.

Example 3

Show that $v = 3x^2y - y^3$ is a harmonic conjugate of $u = x^3 - 3xy^2$.

We simply notice that if $f(z) = z^3$, then f is entire and by the binomial theorem

$$\begin{aligned} f(x+iy) &= (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3 \\ &= \underbrace{\left(x^3 - 3xy^2\right)}_{u} + i \underbrace{\left(3x^2y - y^3\right)}_{v}. \end{aligned}$$

Example 4

Find a harmonic conjugate for $u = \arctan(y/x)$, $x > 0$.

For $x > 0$, $\arctan(y/x) = \operatorname{Arg}(x+iy)$. Let $f(z) = \operatorname{Log} z$.

Then $-if(z)$ is analytic for $x > 0$ and is given by

$$-i\,\mathrm{Log}\,z = -i(\ln|z| + i\,\mathrm{Arg}\,z) = \underbrace{\mathrm{Arg}\,z}_{u} - i\ln|z|.$$

Therefore a harmonic conjugate of $u = \mathrm{Arg}\,z$ is

$$\boxed{v = -\ln|z| = -\frac{1}{2}\ln\left(x^2 + y^2\right).}$$

Harmonic conjugates are *almost* unique. To prove this we require a preliminary result.

Lemma 1

Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ analytic. If the image of f is contained in a line, then f is constant.

Proof. Let $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, and suppose $f(\Omega)$ is a subset of the line

$$aX + bY = c, \ \ (a,b) \neq (0,0).$$

Then $au(x,y) + bv(x,y) = c$ for all $(x,y) \in \Omega$. Differentiating and applying the C-R equations we obtain

$$\left.\begin{array}{rcl} au_x + bv_x & = & 0, \\ au_y + bv_y = -av_x + bu_x & = & 0, \end{array}\right\} \Rightarrow \begin{pmatrix} u_x & v_x \\ -v_x & u_x \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

throughout Ω. Because $(a,b) \neq (0,0)$,

$$0 = \det \begin{pmatrix} u_x & v_x \\ -v_x & u_x \end{pmatrix} = u_x^2 + v_x^2 = |f'(z)|^2 \ \Rightarrow \ f'(z) = 0,$$

everywhere in Ω. As we have seen, this implies that f is constant. □

Theorem 2

Let $\Omega \subset \mathbb{R}^2$ be a domain and suppose u is harmonic on Ω. If v_1 and v_2 are harmonic conjugates of u on Ω, then there is an $a \in \mathbb{R}$ so that $v_1 = v_2 + a$.

Proof. Let $f_j = u + iv_j$ for $j = 1, 2$. Then f_1, f_2 are analytic on Ω. Therefore $f_1 - f_2 = i(v_1 - v_2)$ is analytic and purely imaginary on Ω. By the lemma, $f_1 - f_2$ is constant. The result follows. □

This addresses the uniqueness of harmonic conjugates. What about existence?

Example 5

Does $u = \ln\left(x^2 + y^2\right)$ have a harmonic conjugate on $\mathbb{C}^\times$?

Assume $f = u + iv$ is analytic on $\mathbb{C}^\times$. Then so is $f/2$.

Notice that $\text{Re}(f/2) = u/2 = \ln|z| = \text{Re}(\text{Log}\, z)$ on $\mathbb{C} \setminus (-\infty, 0]$.

Therefore both $v/2$ and $\text{Arg}\, z$ are conjugates of $u/2$ on the slit plane.

Thus $\text{Arg}\, z = a + v/2$ for some $a \in \mathbb{R}$.

But $a + v/2$ extends continuously to $\mathbb{C}^\times$, whereas $\text{Arg}\, z$ does not.

So u *cannot* have a conjugate on $\mathbb{C}^\times$. □

It turns out the problem in the preceding example is fact that $\mathbb{C}^\times$ is *multiply connected*.

Theorem 3

Let $\Omega \subset \mathbb{C}$ be a domain and let $u : \Omega \to \mathbb{R}$ be harmonic. If Ω is simply connected, then u has a harmonic conjugate on Ω.

Sketch of Proof. Fix $P_0 \in \Omega$, let $\omega = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$ and for $P \in \Omega$ define

$$v(P) = \int_{P_0}^{P} \omega,$$

where the integral is taken along any piecewise smooth curve in Ω from P_0 to P.

We first claim that v is *well-defined*, i.e. is **path-independent.**

Let C_1, C_2 be paths in Ω from P_0 to P. Then $C_1 - C_2$ is a loop in Ω. Let R be the region enclosed by $C_1 - C_2$.

Because Ω is simply connected, $R \subset \Omega$. Green's theorem then implies

$$\begin{aligned} \int_{C_1 - C_2} \omega = \iint_R d\omega &= \iint_R \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)\, dA \\ &= \iint_R \Delta u \, dA = 0, \end{aligned}$$

since u is harmonic. Thus

$$\int_{C_1} \omega = \int_{C_2} \omega,$$

and v is well-defined.

To differentiate v at $P = (x, y) \in \Omega$, let $P' = (x + \Delta x, y)$.

Let C be any path in Ω from P_0 to P and let $C' = C + L$, where L is the horizontal segment from P to P'.

Then

$$\begin{aligned} v_x(x, y) &= \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_L \omega \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_0^{\Delta x} -u_y(x + t, y)\, dt \\ &= \lim_{\Delta x \to 0} -u_y(x + h, y) \end{aligned}$$

for some $h \in [0, \Delta x]$, by the Mean Value Theorem.

Since $h \to 0$ as $\Delta x \to 0$, and u_y is continuous, we find that

$$v_x(x, y) = -u_y(x, y).$$

By instead using a vertical segment one can compute $v_y(x, y)$ in a similar manner. The result is

$$v_y(x, y) = u_x(x, y).$$

Thus, v is a harmonic conjugate of u on Ω. $\square$

Remarks.

1. Using a different base point P_1 yields a conjugate v_1 that differs from v by the additive constant $\int_{P_0}^{P_1} \omega$.
2. Strictly speaking, Green's theorem only applies to *simple* closed curves, a hypothesis we cannot assume for $C_1 - C_2$.
3. A rigorous proof applies Green's theorem locally, to subdivisions of the homotopy between C_1 and C_2 in Ω.

Corollary 1

Harmonic conjugates always exist locally.

Assuming the C^∞ nature of analytic functions, we have the following result as well.

Corollary 2

Let $\Omega \subset \mathbb{R}^2$ be a domain. If $u \in C^2(\Omega)$ and $\Delta u = 0$ on Ω, then $u \in C^\infty(\Omega)$.

Proof. Let $P \in \Omega$. Choose an open disk $D \subset \Omega$ containing P.

Since D is simply connected and u is harmonic on D, there is an analytic function $f : D \to \mathbb{C}$ so that $u = \operatorname{Re} f$.

It follows that $u \in C^\infty(D)$. Since $P \in \Omega$ was arbitrary, we conclude that $u \in C^\infty(\Omega)$. $\square$