Integration I

A Review of Line Integrals

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Complex Variables

Paths

Definition

Let $P, Q \in \mathbb{R}^2$. A *path* from P to Q is a continuous function $\gamma : [a, b] \to \mathbb{R}^2$ satisfying $\gamma(a) = P$ and $\gamma(b) = Q$. A path γ is called *simple* if $s \neq t$ implies $\gamma(s) \neq \gamma(t)$.

Definition

A path from a point $P \in \mathbb{R}^2$ to itself is called a *loop*. A loop γ is called *simple* if $\gamma(s) \neq \gamma(t)$ unless $s = t$ or $s, t \in \{a, b\}$.

Remarks.

1. Loops are also called *closed paths*.
2. A loop can also be viewed as a continuous map $\gamma : S^1 \to \mathbb{R}^2$.
3. The image of a path inherits an *orientation* from the linear ordering on $[a, b]$.

Examples

Example 1

The line segment from P to Q is given by $\gamma(t) = (1-t)P + tQ$ with $t \in [0, 1]$.

Example 2

The circle of radius R centered at $P_0 = (x_0, y_0)$ is given by $\gamma(t) = (x_0 + R\cos t, y_0 + R\sin t)$ with $t \in [0, 2\pi]$. This orients the circle counterclockwise.

Example 3

The path $\gamma(t) = (t\cos t, t\sin t)$ is a spiral, since in polar coordinates it is described by $r = t = \theta$.

Reparametrizations

Definition

A *reparametrization* of $[a, b]$ is a strictly increasing continuous bijection $\phi : [\alpha, \beta] \to [a, b]$

Definition

If $\gamma : [a, b] \to \mathbb{R}^2$ is a path from P to Q, and ϕ is a reparametrization of $[a, b]$, we call $\gamma \circ \phi$ a *reparametrization* of γ.

"Being a reparametrization of" is an equivalence relation on the set of paths from P to Q. So we identify a path with its reparametrizations.

All reparametrizations of γ have the same image with the same orientation.

Smooth Paths

Definition

A path $\gamma(t) = (x(t), y(t))$ is said to be *of class* C^k provided $x(t)$ and $y(t)$ are C^k functions.

Definition

A path $\gamma : [a, b] \to \mathbb{R}^2$ is *piecewise* C^k if there exist a partition $\pi = \{t_0 < t_1 < \cdots < t_n\}$ of $[a, b]$ so that the restrictions $\gamma_j = \gamma|_{[t_{j-1}, t_j]}$ are C^k for $j = 1, 2, \ldots, n$.

Remarks.

1. Piecewise smoothness allows us to include (finitely many) discontinuities of the derivative(s) of a path (e.g. polygons).
2. Smoothness is preserved by *smooth* reparametrizations.

Tangent Vectors

Every path $\gamma : [a, b] \to \mathbb{R}^2$ is a curve (in the previous sense) on (a, b).

So, when $\gamma(t) = (x(t), y(t))$ is C^1 we have the *tangent vector* $\gamma'(t) = (x'(t), y'(t))$, given by

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t + h) - \gamma(t)}{h},$$

for $a < t < b$. The *one-sided tangent vector* at $\gamma'(a)$ is computed by instead taking $h \to 0^+$, while $\gamma'(b)$ uses $h \to 0^-$.

The orientation of the image of γ is determined by γ'.

Definition

A loop is said to be *of class* C^0. In general, we say a loop γ is *of class* C^k provided γ' is a class C^{k-1} loop.

Line Integrals

Definition

Let $\Omega \subset \mathbb{R}^2$ be a domain. A *(differential) 1-form* on Ω is a formal linear combination $\omega = P\,dx + Q\,dy$, where $P, Q : \Omega \to \mathbb{R}$. We say ω is C^k provided $P, Q \in C^k(\Omega)$.

1-forms are also known as *vector fields*.

Definition

Let $\omega = P\,dx + Q\,dy$ be a continuous 1-form on a domain $\Omega \subset \mathbb{R}^2$ and let $\gamma : [a, b] \to \Omega$ be a C^1 path. We define the *line integral of ω along γ* to be

$$\int_\gamma \omega = \int_a^b P(\gamma(t))x'(t) + Q(\gamma(t))y'(t)\,dt,$$

where $\gamma(t) = (x(t), y(t))$.

Remarks

1. To evaluate $\int P\,dx + Q\,dy$ we make the formal "substitutions" $x = x(t)$, $y = y(t)$, so that $dx = x'(t)\,dt$, $dy = y'(t)\,dt$.
2. If we let $\mathbf{F} = (P, Q)$ and $\mathbf{r} = (x, y)$, then

$$\int_a^b P(\gamma(t))x'(t) + Q(\gamma(t))y'(t)\,dt = \int_a^b \mathbf{F} \cdot \underbrace{\gamma'(t)\,dt}_{d\mathbf{r}} = \int_\gamma \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_a^b \frac{\mathbf{F} \cdot \gamma'(t)}{|\gamma'(t)|} \underbrace{|\gamma'(t)|\,dt}_{ds = |d\mathbf{r}|} = \int_\gamma \mathrm{proj}_{\mathbf{T}}\,\mathbf{F}\,ds$$

   where $\mathbf{T} = \gamma'(t)/|\gamma'(t)|$ is the unit tangent.
3. $ds = |d\mathbf{r}| = |\gamma'(t)|\,dt = \sqrt{x'(t)^2 + y'(t)^2}\,dt$ is called the *arc length differential*. The *arc length* of γ is $\int_\gamma ds = \int_a^b |\gamma'(t)|\,dt$.

Example 4

Compute $\int_\gamma \frac{-y\,dx+x\,dy}{x^2+y^2}$, where γ is any circle centered at the origin, oriented counterclockwise.

Solution. We parametrize γ by

$$\gamma(t) = (R\cos t, R\sin t), \quad t \in [0, 2\pi].$$

Then $dx = -R\sin t\,dt$, $dy = R\cos t\,dt$ and $x^2 + y^2 = R^2$. Thus

$$\int_\gamma \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{(-R\sin t)(-R\sin t) + (R\cos t)(R\cos t)}{R^2}\,dt$$
$$= \int_0^{2\pi} dt = \boxed{2\pi}.$$

□

Remark. It follows that if we orient the circle clockwise, we obtain -2π.

Definition

Let ω be a continuous 1-form of on a domain $\Omega \subset \mathbb{R}^2$ and let γ be a piecewise C^1 path in Ω, with C^1 partitions $\gamma_1, \gamma_2, \ldots, \gamma_n$. The *line integral of* ω *along* γ is defined to be

$$\int_\gamma \omega = \sum_{j=1}^{n} \int_{\gamma_j} \omega.$$

Remarks.

1. One can argue that any refinement of a given C^1 partition does not affect the value of the integral.
2. As any two partitions always have a common refinement, the value of the integral is independent of the C^1 partition used.
3. It is easy to see that $\int_\gamma \omega$ is invariant under a smooth reparametrization ϕ: just substitute $u = \phi(t)$.

Properties of Line Integrals

Linearity of the definite integral implies that if ω_1 and ω_2 are 1-forms and $\alpha, \beta \in \mathbb{R}$, then

$$\boxed{\int_\gamma \alpha\omega_1 + \beta\omega_2 = \alpha \int_\gamma \omega_1 + \beta \int_\gamma \omega_2.}$$

If $(-\gamma)(t) = \gamma(b - t)$ for $t \in [0, b - a]$ (the *opposite* path), then

$$\boxed{\int_{-\gamma} \omega = -\int_\gamma \omega.}$$

This follows upon making the substitution $u = b - t$ in $\int_{-\gamma} \omega$.

Properties (Continued)

If $\gamma : [a, b] \to \mathbb{R}^2$ is a path from P to Q and $\kappa : [c, d] \to \mathbb{R}^2$ is a path from Q to R, then their *concatenation* is the path given by

$$(\gamma + \kappa)(t) = \begin{cases} \gamma(a + t(b - a)) & \text{for } t \in [0, 1], \\ \kappa(c + (t - 1)(d - c)) & \text{for } t \in [1, 2]. \end{cases}$$

The concatenation of piecewise C^1 paths is again piecewise C^1, and the definition of the integral immediately implies

$$\boxed{\int_{\gamma+\kappa} \omega = \int_\gamma \omega + \int_\kappa \omega.}$$

Derivatives of Forms

A real-valued function $f(x, y)$ is called a *0-form*.

The *(exterior) derivative* of a 0-form f is the 1-form

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy.$$

The *(exterior) derivative* of a 1-form $\omega = P\,dx + Q\,dy$ is the *2-form*

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA,$$

where $dA = dx\,dy$.

Remarks.

1. The derivative of a 0-form is also known as the *gradient*.
2. The derivative of a 1-form is also called the *curl*.

The Fundamental Theorem of Calculus

Theorem 1 (FTOC for Line Integrals)

Let $\Omega \subset \mathbb{R}^2$ be a domain and let $f \in C^1(\Omega)$. For any $P, Q \in \Omega$ and any piecewise C^1 path γ in Ω from P to Q one has

$$\int_\gamma df = f(Q) - f(P).$$

Proof. We may assume $\gamma(t) = (x(t), y(t))$ is C^1. The general result follows from a "telescoping" argument.
The multivariate chain rule and single-variable FTOC then imply

$$\begin{aligned}\int_\gamma df &= \int_a^b f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)\,dt \\ &= \int_a^b \frac{d}{dt} f(\gamma(t))\,dt = f(\gamma(b)) - f(\gamma(a)) = f(Q) - f(P).\end{aligned}$$

□

Exact Forms and Path-Independence

Definition

A 1-form ω is called *exact* if $\omega = df$ for some 0-form f.

FTOC implies that line integrals of exact forms are *path-independent*.

In fact, path-independence characterizes exact 1-forms.

Theorem 2

Let $\Omega \subset \mathbb{R}^2$ be a domain and let ω be a C^0 1-form on Ω. The following are equivalent:

1. *ω is exact.*
2. *$\int_\gamma \omega$ is path-independent in Ω, i.e. depends only on the endpoints of γ.*
3. *$\int_\gamma \omega = 0$ for all* closed *piecewise C_1 paths in Ω.*

Sketch of Proof

1. $\Rightarrow$ 3. Follows from FTOC.

3. $\Rightarrow$ 2. Let γ_1, γ_2 be closed piecewise C^1 paths in Ω from P to Q. Then $\gamma = \gamma_1 - \gamma_2$ is closed, so that

$$0 = \int_\gamma \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega,$$

by hypothesis. Thus $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$.

2. $\Rightarrow$ 1. Fix $P_0 \in \Omega$ and for $P \in \Omega$ define $f(P) = \int_{P_0}^{P} \omega$, the integral being taken over any piecewise C^1 path in Ω from P_0 to P. This is well-defined by hypothesis.

One can compute f_x and f_y by an earlier argument, and concludes that $df = \omega$. □

Closed and Exact Forms

Definition

A 1-form ω is called *closed* if $d\omega = 0$.

If $\Omega \subset \mathbb{R}^2$ is a domain and $f \in C^2(\Omega)$, then by Clairaut's theorem:

$$d^2 f = d(df) = d\left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\right) = \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\,dA = 0.$$

Therefore:

Theorem 3

Every exact form is closed.

Remark: Not all closed forms on a domain Ω are exact (see below). The quotient space (closed 1-forms)/(exact 1-forms) is the *first de Rham cohomology group* of Ω.

A Non-Exact Closed 1-Form

Consider the 1-form

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

on $\mathbb{R}^2 \setminus \{(0,0)\}$. Its derivative is

$$\begin{aligned} d\omega &= \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} \right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2} \right) \right) dA \\ &= \left(\frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} + \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} \right) dA = 0, \end{aligned}$$

so ω is closed. But as we have seen,

$$\int_{S^1} \omega = 2\pi \neq 0,$$

so that ω *cannot* be exact.