Integration III

Complex Integration

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Complex Variables

Complex Differentials

Let $f(z)$ be a complex-valued function defined on a domain $\Omega \subset \mathbb{C}$.

To define complex line integrals in Ω we first need to define the complex 1-form $f(z)\,dz$.

Let $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ so that $f = u + i\,v$.

We define $dz = dx + i\,dy$ and proceed formally:

$$f(z)\,dz = (u + i\,v)(dx + i\,dy) := (u\,dx - v\,dy) + i(v\,dx + u\,dy).$$

Given a path γ in Ω we therefore define

$$\int_\gamma f(z)\,dz = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy. \qquad (1)$$

Another Approach

If $h(t)$ is a complex-valued function on an interval $t \in [a, b]$, we define

$$\int_a^b h(t)\, dt = \int_a^b \operatorname{Re} h(t)\, dt + i \int_a^b \operatorname{Im} h(t)\, dt.$$

If a path is parametrized by $\gamma(t)$, $t \in [a, b]$, it is also natural to make the formal substitution $z = \gamma(t)$, $dz = \gamma'(t)\, dt$ and define

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt. \qquad (2)$$

One can show that definitions (1) and (2) are equivalent.

Definition (1) is primarily of theoretical interest. Definition (2) is usually used in practice.

An Important Example

Example 1

Let $z_0 \in \mathbb{C}$. Evaluate $\displaystyle\int_C \frac{dz}{z - z_0}$, where C is the circle $|z - z_0| = r$, oriented positively.

Solution. We parametrize C by $\gamma(\theta) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$.

Setting $z = \gamma(\theta)$, we have $dz = \gamma'(\theta)\, d\theta = ire^{i\theta}\, d\theta$, so that

$$\int_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{ire^{i\theta}\, d\theta}{(z_0 + re^{i\theta}) - z_0} = \int_0^{2\pi} i\, d\theta = \boxed{2\pi i}.$$

□

Another Example

Example 2

Evaluate $\int_C z^2\,dz$, where C is the line segment from i to 1.

Solution. We parametrize C as $\gamma(t) = t + (1-t)i$, $t \in [0,1]$.

If $z = \gamma(t)$, then $dz = 1 - i\,dt$ so that

$$\begin{aligned}\int_C z^2\,dz &= \int_0^1 (t + (1-t)i)^2(1-i)\,dt \\ &= \int_0^1 -2t^2 + 4t - 1 + i(1-2t^2)\,dt \\ &= \int_0^1 -2t^2 + 4t - 1\,dt + i\int_0^1 1 - 2t^2\,dt\end{aligned}$$

$$= -\frac{2}{3}t^3 + 2t^2 - t\Big|_0^1 + i\left(t - \frac{2}{3}t^3\Big|_0^1\right) = \frac{1}{3} + \frac{i}{3} = \boxed{\frac{1+i}{3}}.$$

□

Remarks.

1. The first example will be relevant to our discussion of *residues*.
2. The second example can be explained by an appropriate version of the *Fundamental Theorem of Calculus*.

Properties of Complex Integrals

Let $h(t), j(t)$ be complex-valued functions for $t \in [a, b]$ and let $\alpha, \beta \in \mathbb{C}$.

One can show directly from the definition (HW) that

$$\int_a^b \alpha h(t) + \beta j(t)\, dt = \alpha \int_a^b h(t)\, dt + \beta \int_a^b j(t)\, dt,$$

i.e. the real-complex integral is *linear*.

It follows from (1) and (2) that complex line integrals are also linear:

$$\int_\gamma \alpha f(z) + \beta g(z)\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz.$$

Definition (1) also implies that the integral is path-additive:

$$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz,$$

where $\gamma_1 + \gamma_2$ denotes the concatenation of γ_1 and γ_2.

And reversing directions negates the integral:

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz.$$

Moral. The complex line integral enjoys the usual properties of line integrals of (real-valued) 1-forms.

A Special Upper Bound

Recall that if $h(t)$ is *real-valued* on $[a, b]$, then we have the upper bound

$$\left|\int_a^b h(t)\,dt\right| \le \int_a^b |h(t)|dt.$$

We will show the same result still holds if $h(t)$ is *complex-valued*.

Lemma 1

Let $h(t)$ be complex-valued on $[a, b]$. Then

$$\left|\int_a^b h(t)\,dt\right| \le \int_a^b |h(t)|dt.$$

We can use the linearity of the integral to continue to eschew the use of Riemann sums in our proof.

Proof of Lemma 1

Write $\int_a^b h(t)\,dt = Re^{i\phi}$ with $R \geq 0$. Then

$$\begin{aligned} R = \operatorname{Re} R &= \operatorname{Re}\left(e^{-i\phi}\int_a^b h(t)\,dt\right) \\ &= \operatorname{Re}\int_a^b e^{-i\phi}h(t)\,dt = \int_a^b \operatorname{Re}\left(e^{-i\phi}h(t)\right)\,dt \\ &\leq \int_a^b \left|e^{-i\phi}h(t)\right|\,dt = \int_a^b |h(t)|\,dt, \end{aligned}$$

which is what we needed to show. □

General Bounds

We will use Lemma 1 to prove the following fundamental upper bounds on complex line integrals.

Theorem 1

Let $\Omega \subset \mathbb{C}$ be a domain, $f : \Omega \to \mathbb{C}$ be continuous, and γ be a piecewise C^1 path in Ω. Suppose $|f| \leq M$ on γ and let L be the arc length of γ. We then have

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)|\,|dz| \leq ML.$$

Here $|dz| = |dx + i\,dy| = \sqrt{dx^2 + dy^2} = ds$ is the *arc length differential*.

If $z = \gamma(t)$ then $dz = \gamma'(t)\,dt$ and $|dz| = |\gamma'(t)|\,dt$.

Proof of Theorem 1

Because of path-linearity, it suffices to assume γ is parametrized by $\gamma(t)$, $t \in [a, b]$.

By Lemma 1 we then have

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right| \le \int_a^b \underbrace{|f(\gamma(t))|}_{|f(z)|} \cdot \underbrace{|\gamma'(t)|\,dt}_{|dz|}$$

$$\le \int_a^b M\,|\gamma'(t)|\,dt = M\int_a^b \underbrace{|\gamma'(t)|\,dt}_{|dz|=ds} = ML.$$

The result follows. □

Example

Here's a typical application of the *ML*-estimate of Theorem 1.

Example 3

Use an *ML*-estimate to show that $\lim_{r\to\infty} \int_{S_r} \frac{z\,dz}{z^3+1} = 0$, where S_r is the semicircle $|z| = r$, $\operatorname{Im} z \geq 0$, oriented positively.

Solution. On S_r we have $|z| = r$ and, provided $r > 1$,

$$|z^3 + 1| \geq |z|^3 - 1 = r^3 - 1 > 0.$$

Thus

$$\left|\frac{z}{z^3+1}\right| < \frac{r}{r^3-1} = M$$

on S_r.

Since $L = \pi r$ is the length of S_r, Theorem 1 implies that

$$\left| \int_{S_r} \frac{z\,dz}{z^3+1} \right| \leq \frac{r}{r^3-1} \cdot \pi r = \frac{\pi r^2}{r^3-1}.$$

Calc. I implies that

$$\frac{\pi r^2}{r^3-1} \to 0 \quad \text{as } r \to \infty.$$

So by the Squeeze Theorem from Calc I.,

$$\lim_{r\to\infty} \int_{S_r} \frac{z\,dz}{z^3+1} = 0.$$

□