Integration II

Green's Theorem

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Complex Variables

Last Time

We defined the integral of a 1-form (vector field) $\omega = P\,dx + Q\,dy$ along a smooth path $\gamma(t) = (x(t), y(t))$ with parameter domain $[a, b]$ to be

$$\int_\gamma \omega = \int_a^b P(\gamma(t))x'(t) + Q(\gamma(t))y'(t)\,dt,$$

and extended this definition to piecewise smooth paths additively.

According to FTOC, for any $f(x, y)$ and any path γ from P to Q:

$$\int_\gamma df = f(Q) - f(P),$$

where $df = f_x\,dx + f_y\,dy$ (the gradient).

Example

Example 1

Evaluate $\int_\gamma (x+y)\,dx + xy\,dy$, where γ is the boundary of the unit square $[0,1]\times[0,1]$, oriented counterclockwise.

Solution. We integrate along each edge separately, then add the results. Let $\omega = (x+y)\,dx + xy\,dy$.

γ_1 $(y=0)$: Here $dy = 0$ so that

$$\int_{\gamma_1}\omega = \int_0^1 x dx = \frac{1}{2}.$$

γ_2 $(x=1)$: Here $dx = 0$ so that

$$\int_{\gamma_2}\omega = \int_0^1 y\,dy = \frac{1}{2}.$$

$\underline{\gamma_3\ (y=1)}$: Here $dy = 0$ so that

$$\int_{\gamma_3} \omega = -\int_0^1 (x+1)\, dx = -\frac{3}{2}.$$

$\underline{\gamma_4\ (x=0)}$: Here $dx = 0$ so that

$$\int_{\gamma_1} \omega = \int_0^1 0\, dy = 0.$$

Thus

$$\int_{\gamma} \omega = \frac{1}{2} + \frac{1}{2} - \frac{3}{2} + 0 = \boxed{-\frac{1}{2}.}$$

Closed and Exact Forms

A 1-form $\omega = P\,dx + Q\,dy$ is called *exact* provided $\omega = df = f_x\,dx + f_y\,dy$ for some $f(x,y)$ (ω has an "antiderivative"). It is called *closed* provided $d\omega = (Q_x - P_y)\,dA = 0$.

FTOC tells us:

- how to integrate exact 1-forms;
- integrals of exact 1-forms are *path-independent*.

We showed that path-independence classifies the exact 1-forms.

Because $d^2 f = 0$ for all $f(x,y)$ (Clairaut's theorem), exact forms are closed. The converse is *false* in general, due to the connectivity of the domain of integration.

Green's theorem:

- generalizes FTOC to (possibly) inexact 1-forms;
- quantifies the failure of closed forms to be exact.

Green's Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain.

We say Ω has a *piecewise C^k boundary* provided the boundary $\partial\Omega$ is the union of finitely many simple closed piecewise C^k paths.

In this case we say $\partial\Omega$ is *positively oriented* if Ω is always to the left of (the tangent vector to) $\partial\Omega$.

Theorem 1 (Green's Theorem)

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with positively oriented piecewise C^1 boundary $\partial\Omega$. If ω is a C^1 1-form on (a neighborhood of) $\Omega \cup \partial\Omega$, then

$$\int_{\partial\Omega} \omega = \iint_{\Omega} d\omega.$$

Remarks

1. Roughly speaking, the "outside edge" of Ω is oriented counterclockwise, while any "inner edges" or "holes" are oriented clockwise.

2. Non-simple boundaries can be handled by subdividing and reorienting (if necessary).

3. Using Green's theorem one can derive the formula

$$A(\Omega) = \frac{1}{2} \int_{\partial\Omega} -y\,dx + x\,dy$$

for the area of Ω in terms of its boundary.

Examples

Example 2

Use Green's theorem to reevaluate $\gamma(x+y)\,dx + xy\,dy$ along the positively oriented boundary of the unit square.

Solution. If $\omega = (x+y)\,dx + xy\,dy$, then $d\omega = (y-1)\,dA$. Thus

$$\int_\gamma \omega = \int_0^1 \int_0^1 y - 1\,dx\,dy = \int_0^1 y - 1\,dy = \frac{1}{2} - 1 = \boxed{-\frac{1}{2}}.$$

Example 3

Integrate $\omega = x^2\,dy$ around the boundary of the portion of the unit disk in the first quadrant.

Solution. We have $d\omega = 2x\,dA$ so that

$$\int_{\partial\Omega} \omega = \iint_\Omega 2x\,dA = \int_0^{\pi/2} \int_0^1 2r\cos\theta\,r dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta \, d\theta \int_0^1 r^2 \, dr = \frac{1}{3} \sin\theta \Big|_0^{\pi/2} = \boxed{\frac{1}{3}.}$$

Example 4

Let γ be any simple closed piecewise C^1 path enclosing the origin, oriented positively. Prove that $\int_\gamma \frac{-y\,dx + x\,dy}{x^2+y^2} = 2\pi$.

Solution. The function $\sqrt{x^2+y^2}$ is a continuous function on the compact set $C = \gamma([a,b])$, so attains its absolute minimum value ϵ. Since $(0,0) \notin C$, $\epsilon > 0$.

It follows that the disk $\sqrt{x^2+y^2} < \epsilon$ is contained within C. Let C' denote the circle $x^2+y^2 = (\epsilon/2)^2$, oriented counterclockwise. Let Ω be the region between C and C'.

Let $\omega = \frac{-y\,dx + x\,dy}{x^2+y^2}$. We have seen that $\int_{C'} \omega = 2\pi$ and $d\omega = 0$.

By Green's theorem

$$0 = \iint_{\Omega} 0\,dA = \int_{\partial\Omega} \omega = \int_{\gamma} \omega - \int_{C'} \omega.$$

The result follows. □

Similar reasoning can be used to show that if Ω has "holes," then the loop integral of any closed 1-form ω in Ω reduces to the sum of smaller loop integrals, one around each "hole." These are the *periods* of ω.

The periods of closed forms are precisely what prevents them from being exact!

Simply Connected Domains

A simply connected domain Ω is "hole-free."

Hence closed forms have *no periods* and therefore their loop integrals in Ω *vanish*.

It follows that every closed form is exact.

Theorem 2

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. Then every closed C^1 1-form on Ω is exact.

This subsumes our earlier work with harmonic functions.

Harmonic Functions Revisited

Recall that $u \in C^2(\Omega)$ is *harmonic* provided $\Delta u = 0$, and that v is a *harmonic conjugate* of u provided the C-R equations hold $(u_x = v_y,\ u_y = -v_x)$.

Given a harmonic u on Ω, the 1-form $\omega = -u_y\,dx + u_x\,dy$ is C^1 and closed since

$$d\omega = \left((u_x)_x - (-u_y)_y\right)\,dA = \Delta u\,dA = 0.$$

If Ω is simply connected, then ω must be exact, by Theorem 2.

So there is a $v \in C^1(\Omega)$ so that $dv = \omega$, i.e.

$$v_x\,dx + v_y\,dy = -u_y\,dx + u_x\,dy \quad \Rightarrow \quad \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases},$$

so that v is a harmonic conjugate of u on Ω!

Now suppose Ω contains the open disk $\{|z - z_0| < R\}$ and u is harmonic on Ω. Let $0 < r < R$.

Because $\omega = -u_y\,dx + u_x\,dy$ is closed, Green's theorem implies

$$0 = \iint_{|z-z_0|\leq r} 0\,dA = \int_{|z-z_0|=r} \omega.$$

Parametrize the circle by $\gamma(\theta) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$ to get

$$\begin{aligned}
\int_{|z-z_0|=r} \omega &= \int_0^{2\pi} -u_y(\gamma(\theta))\,(-r\sin\theta) + u_x(\gamma(\theta))\,(r\cos\theta)\,d\theta \\
&= r\int_0^{2\pi} u_x(\gamma(\theta))\cos\theta + u_y(\gamma(\theta))\sin\theta\,d\theta \\
&= r\int_0^{2\pi} \frac{\partial}{\partial r}u(\gamma(\theta))\,d\theta = r\frac{\partial}{\partial r}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta.
\end{aligned}$$

We conclude that $\frac{\partial}{\partial r}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta = 0$ so that

$$\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta = C$$

for $0<r<R$.

To evaluate C let $r\to 0^+$:

$$C = \lim_{r\to 0^+}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta = \int_0^{2\pi}\lim_{r\to 0^+} u(z_0+re^{i\theta})\,d\theta$$
$$= \int_0^{2\pi} u(z_0)\,d\theta = 2\pi u(z_0).$$

This proves that

$$\boxed{u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta.}$$

Summary

Theorem 3

Let u be harmonic on a domain Ω containing the open disk $\{|z - z_0| < R\}$. Then u has the mean value property, *namely*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

for $0 < r < R$.

Remarks.

- The mean value property leads to the so-called *Maximum Principle*.
- The interchange of derivative and limit with the integral can be carefully justified.

Mean Value on a Circle

Given a path γ and a function $f(x, y)$, the *mean value* of f on γ is

$$\frac{1}{\ell(\gamma)} \int_\gamma f\, ds,$$

where $ds = |\gamma'(t)|\, dt$ and $\ell(\gamma)$ is the arc length of γ.

If $\gamma(\theta) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$, then $\gamma'(\theta) = ire^{i\theta}$ and $ds = r\, d\theta$.

So the average value of $f(x, y)$ on the circle $|z - z_0| = r$ is

$$\frac{1}{2\pi r} \int_0^{2\pi} f(z_0 + re^{i\theta}) r\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

The mean value property therefore states that *the value of a harmonic function at a point z_0 is equal to the average of its values on any circle centered at z_0.*