Integration IV

FTOC for Analytic Functions

Ryan C. Daileda

Trinity University

Complex Variables

Earlier Versions of FTOC

We've seen the Fundamental Theorem of Calculus (FTOC) twice.

Theorem 1 (FTOC for Real-Variable Functions)

If $f(t)$ is continuous on $[a, b]$ and $F'(t) = f(t)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Theorem 2 (FTOC for 1-Forms)

If $\omega = P\,dx + Q\,dy$ is a continuous 1-form on a domain $\Omega \subset \mathbb{R}^2$ and $\omega = df$ for some C^1 function $f : \Omega \to \mathbb{R}$ (ω is exact), then for any path γ in Ω from P to Q one has

$$\int_\gamma \omega = f(Q) - f(P).$$

Remarks

1. FTOC for functions states that if a function has an antiderivative, then the value of the integral *depends only on the values of the antiderivative at the endpoints of the interval of integration.*

2. FTOC for 1-forms states that if a 1-form is a gradient, then the value of the integral *depends only on the values of the potential function at the endpoints of the path of integration.*

3. We will prove a version of FTOC for complex functions that combines both of these ideas.

FTOC for Analytic Functions

We now state and prove our main result.

Theorem 3 (FTOC for Analytic Functions, Part I)

Let $\Omega \subset \mathbb{C}$ be a domain. Suppose $f : \Omega \to \mathbb{C}$ is continuous, $F : \Omega \to \mathbb{C}$ is analytic, and $F'(z) = f(z)$ throughout Ω. Then for any path γ in Ω from z_1 to z_2 one has

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1).$$

Proof. Let $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, and recall that

$$\int_\gamma f(z)\,dz = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy := \int_\gamma \omega_1 + i\int_\gamma \omega_2.$$

Let $U = \operatorname{Re} F$ and $V = \operatorname{Im} F$.

Since F is analytic on Ω, the C-R equations hold:

$$U_x = V_y \quad \text{and} \quad U_y = -V_x.$$

Moreover

$$u + iv = f(z) = F'(z) = U_x + iV_x = V_y - iU_y.$$

Hence

$$\omega_1 = u\,dx - v\,dy = U_x\,dx + U_y\,dy = dU,$$

and

$$\omega_2 = v\,dx + u\,dy = V_x\,dx + V_y\,dy = dV.$$

By FTOC for 1-forms we therefore have

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_\gamma \omega_1 + i\int_\gamma \omega_2\\ &= \int_\gamma dU + i\int_\gamma dV\\ &= U(z_2) - U(z_1) + i(V(z_2) - V(z_1))\\ &= F(z_2) - F(z_1).\end{aligned}$$

□

Notice that FTOC implies that when $f(z)$ has a (complex) anti-derivative, $\int_\gamma f(z)\,dz$ is *path-independent*.

Examples

Example 1

Compute $\int_\gamma z^2\,dz$, where γ is any path from i to 1.

Solution. We use FTOC.

The function $F(z) = z^3/3$ is an anti-derivative of $f(z) = z^2$ on $\mathbb{C}$.

By FTOC

$$\int_\gamma z^2\,dz = \left.\frac{z^3}{3}\right|_i^1 = \frac{1^3}{3} - \frac{i^3}{3} = \boxed{\frac{1+i}{3}}.$$

□

Example 2

Use FTOC to evaluate $\displaystyle\oint_{|z|=r} \frac{dz}{z}$.

Solution. The multi-valued function $F(z) = \log z$ is an anti-derivative for $f(z) = 1/z$, but it cannot be defined continuously on C_r because of the necessary branch cut.

So we choose the branch $F(z) = \operatorname{Log} z$ and remove the arc $A_\epsilon = \{|z| = r\} \cap \{|\arg z - \pi| \leq \epsilon\}$.

The complementary "almost circle" C_ϵ avoids the branch cut.

So we will apply FTOC and let $\epsilon \to 0^+$ to "close the circle."

First of all

$$\int_{|z|=r} \frac{dz}{z} = \int_{A_\epsilon} \frac{dz}{z} + \int_{C_\epsilon} \frac{dz}{z},$$

and by an easy *ML*-estimate

$$\left| \int_{A_\epsilon} \frac{dz}{z} \right| \leq \frac{1}{r} \ell(A_\epsilon) = \frac{2\epsilon r}{r} = 2\epsilon.$$

It follows that

$$\begin{aligned}
\int_{|z|=r} \frac{dz}{z} &= \lim_{\epsilon \to 0^+} \int_{|z|=r} \frac{dz}{z} = \lim_{\epsilon \to 0^+} \left(\int_{A_\epsilon} \frac{dz}{z} + \int_{C_\epsilon} \frac{dz}{z} \right) \\
&= \lim_{\epsilon \to 0^+} \int_{C_\epsilon} \frac{dz}{z} = \lim_{\epsilon \to 0^+} \left(\operatorname{Log} z \Big|_{re^{i(\pi+\epsilon)}}^{re^{i(\pi-\epsilon)}} \right) \\
&= \lim_{\epsilon \to 0^+} \Big(\ln r + i(\pi - \epsilon) - (\ln r + i(-\pi + \epsilon)) \Big) \\
&= \lim_{\epsilon \to 0^+} 2(\pi - \epsilon) i = \boxed{2\pi i}.
\end{aligned}$$

FTOC for Analytic Functions...Again

It turns out that on a simply connected domain, every analytic function has an antiderivative.

Thus, FTOC applies whenever the integrand is analytic.

Theorem 4 (FTOC for Analytic Functions, Part II)

Let $\Omega \subset \mathbb{C}$ be a simply connected domain and let $f : \Omega \to \mathbb{C}$ be analytic and C^1. Then there is an analytic $F : \Omega \to \mathbb{C}$ so that $F'(z) = f(z)$ throughout Ω.

Proof. Consider the 1-forms

$$\omega_1 = u\,dx - v\,dy \quad \text{and} \quad \omega_2 = v\,dx + u\,dy.$$

Both ω_1 and ω_2 are closed by the C-R equations:

$$d\omega_1 = (-v_x - u_y)\,dA = (u_y - u_y)\,dA = 0,$$
$$d\omega_2 = (u_x - v_y)\,dA = (u_x - u_x)\,dA = 0.$$

Because Ω is simply connected, both forms are exact:

$$\omega_1 = dU \quad \text{and} \quad \omega_2 = dV.$$

We claim that $F = U + iV$ is analytic on Ω and that $F'(z) = f(z)$.

The C-R equations hold for U and V since

$$U_x = u = V_y \quad \text{and} \quad U_y = -v = -V_x.$$

Moreover, F is C^1 because ω_1 and ω_2 are continuous.

We conclude that F is analytic. Finally, we have

$$F' = U_x + iV_x = u + iv = f,$$

as claimed. □

Suppose $f(z)$ has an antiderivative $F(z)$ in Ω. Fix $z_0 \in \Omega$. Then

$$G(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta \tag{1}$$

is independent of the path from z_0 to z in Ω.

Since $G(z) = F(z) - F(z_0)$, G is also an antiderivative of f.

That the integral formula (1) yields an antiderivative of f is completely analogous to Part 1 of FTOC in Calc. I.

A Preview of Cauchy's Theorem

Cauchy's Theorem is one of the fundamental results in the theory of integration of analytic functions. It has the following simple statement.

Theorem 5 (Cauchy's Theorem)

Let Ω be a bounded domain with piecewise smooth boundary $\partial\Omega$. If $f : \Omega \to \mathbb{C}$ is analytic, C^1, and f extends smoothly to $\partial\Omega$, then

$$\int_{\partial\Omega} f(z)\, dz = 0.$$

We will give our textbook's proof, based on Green's theorem.

Unfortunately, as we will see, this later leads to circular reasoning.

We will avoid this with a different proof next time.

Proof

Let $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

According to Green's theorem and the C-R equations we have

$$\begin{aligned}\int_{\partial\Omega} f(z)\,dz &= \int_{\partial\Omega} u\,dx - v\,dy + i\int_{\partial\Omega} v\,dx + u\,dy \\ &= \iint_{\Omega} (-v_x - u_y)\,dA + i\iint_{\Omega} (u_x - v_y)\,dA \\ &= \iint_{\Omega} 0\,dA + i\iint_{\Omega} 0\,dA \\ &= 0.\end{aligned}$$

□

Trouble

As they stand, our statement and proof of Cauchy's Theorem are technically correct.

Nonetheless, this version is inadequate for our purposes.

The trouble is the C^1 hypothesis, which the book assumes as part of the definition of "analytic."

One of the consequences of Cauchy's Theorem is the Cauchy Integral Formula.

The integral formula can be used to show that every analytic function is infinitely differentiable.

One then concludes that if f' exists, it *must* be continuous, because f'' exists.

So one need not assume C^1 for analytic functions: it's automatic.

Except that we just used the C^1 hypothesis to prove Cauchy's theorem!

So if we assume analytic functions are C^1, we can prove that analytic functions are C^1.

D'oh!