The Maximum Principle

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Complex Variables

Recall

As a consequence of Green's theorem we deduced the following result.

Theorem 1 (Mean Value Property for Harmonic Functions)

Let u be harmonic on a domain Ω containing the open disk $\{|z - z_0| < R\}$. Then u has the mean value property, *namely*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

for $0 < r < R$.

We showed that this says $u(z_0)$ is the average of the values of u on any circle centered at z_0.

The Strict Maximum Principle

The mean value property has the following important consequence.

Theorem 2 (Strict Maximum Principle for Harmonic Functions)

Let $\Omega \subset \mathbb{C}$ be a domain and let $u : \Omega \to \mathbb{R}$ be harmonic. Suppose $u(z) \leq M$ for all $z \in \Omega$. If $u(z_0) = M$ for some $z_0 \in \Omega$, then $u \equiv M$ on Ω.

Moral. A nonconstant harmonic function on a domain cannot attain an absolute maximum value there.

Proof. Let $S = u^{-1}(\{M\}) = \{z \in \Omega \,|\, u(z) = M\}$.

Since u is continuous and $\{M\}$ is closed, $S \subset \Omega$ is closed.

We claim that S is open as well.

Let $z_1 \in S$ and choose $R > 0$ so that the disk $|z - z_1| < R$ is contained in Ω.

For any $0 < r < R$ we have $u(z_1) = \frac{1}{2\pi} \int_0^{2\pi} u(z_1 + re^{i\theta})\, d\theta$, so that

$$\int_0^{2\pi} u(z_1 + re^{i\theta})\, d\theta = 2\pi u(z_1) = \int_0^{2\pi} u(z_1)\, d\theta.$$

Linearity of the integral implies that

$$\int_0^{2\pi} u(z_1) - u(z_1 + re^{i\theta})\, d\theta = 0. \qquad (1)$$

Because $u(z_1 + re^{i\theta}) \leq M = u(z_1)$, the integrand is nonnegative.

The integrand is continuous as well, so (1) implies that it is $\equiv 0$.

That is, $u \equiv u(z_1) = M$ on the circle $|z| = r$.

Because $0 < r < R$ was arbitrary, we conclude that $u \equiv M$ on the disk $|z - z_1| < R$.

So the disk belongs to S, and thus S must be open.

Because Ω is connected and S is clopen, either $S = \varnothing$ or $S = \Omega$.

If $u(z_0) = M$, then $S \neq \varnothing$, and the result follows. □

We can easily extend the maximum principle to analytic functions.

Theorem 3 (Strict Maximum Principle for Analytic Functions)

Let $\Omega \subset \mathbb{C}$ be a domain and suppose $f : \Omega \to \mathbb{C}$ is analytic. If $|f(z)| \leq M$ for all $z \in \Omega$ and $|f(z_0)| = M$, then f is constant on Ω.

Moral. The modulus of a nonconstant analytic function on a domain cannot attain an absolute maximum value there.

Proof. Write $f(z_0) = Me^{i\theta}$ and let $g(z) = e^{-i\theta} f(z)$.

$g(z)$ is analytic and satisfies $|g(z)| \leq M$ for $z \in \Omega$ and $g(z_0) = M$.

Let $u = \operatorname{Re} g$. Then u is harmonic on Ω and satisfies

$$u \leq |u| = |\operatorname{Re} g| \leq |g| \leq M.$$

Because $u \leq M$ and $u(z_0) = \operatorname{Re} g(z_0) = \operatorname{Re} M = M$, the maximum principle implies $u \equiv M$.

Finally,

$$M^2 + (\operatorname{Im} g)^2 = (\operatorname{Re} g)^2 + (\operatorname{Im} g)^2 = |g|^2 \leq M^2$$

implies $\operatorname{Im} g \equiv 0$.

Thus $g \equiv \operatorname{Re} g = u \equiv M$ and $f = e^{i\theta} g \equiv M e^{i\theta}$. □

Absolute Extrema

Harmonic and analytic functions cannot have absolute maxima on connected *open* sets. What does this say about maxima on *compact* sets?

Theorem 4 (Maximum Principle for Harmonic Functions)

Let $\Omega \subset \mathbb{C}$ be a bounded domain with (compact) closure Ω^*. If $u : \Omega \to \mathbb{R}$ is harmonic and extends continuously to Ω^*, then the abs. maximum value of u on Ω^* must occur on the boundary $\partial\Omega$.

Theorem 5 (Maximum Principle for Analytic Functions)

Let $\Omega \subset \mathbb{C}$ be a bounded domain with (compact) closure Ω^*. If $f : \Omega \to \mathbb{C}$ is analytic and extends continuously to Ω^*, then the abs. maximum value of $|f|$ on Ω^* must occur on the boundary $\partial\Omega$.

Proof

We prove the harmonic version.

Because Ω^* is compact, the Extreme Value Theorem guarantees that u has a maximum $u(z_0) = M$ for some $z_0 \in \Omega^*$.

If $z_0 \in \partial\Omega$, we're done. If not, then $z_0 \in \Omega$, and u is constant by the Strict Maximum Principle.

The maximum clearly occurs in $\partial\Omega$ in this case, too, by continuity. □

Remark. The proof of the analytic version is identical. Simply replace u by $|f|$ throughout.

More Remarks

1. By applying the Strict Maximum Principle to $-u$ one obtains the *Strict Minimum Principle for Harmonic Functions*.

2. There is also a *Strict Minimum Principle for Analytic Functions*, obtained by replacing f with $1/f$, but it comes with the additional caveat that f be *zero-free* on Ω.

3. The maximum principle has a number of interesting consequences. We will explore one of them here.

The Fundamental Theorem of Algebra

We can use the maximum principle to prove the following fundamental algebraic result.

Theorem 6 (The Fundamental Theorem of Algebra)

Every nonconstant polynomial with complex coefficients has a root in $\mathbb{C}$.

Remarks.

1. The Fundamental Theorem states that $\mathbb{C}$ is an *algebraically closed* field.
2. Gauss proved the Fundamental Theorem in his doctoral dissertation.

Proof

We proceed by contradiction.

Let $p(z) \in \mathbb{C}[z]$ be nonconstant and suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$.

Let $f(z) = 1/p(z)$. Since p is entire and zero-free, so is f.

Since $\lim_{z \to \infty} p(z) = \infty$ (HW), $\lim_{z \to \infty} f(z) = 0$ (HW).

We claim that $f \equiv 0$, which is impossible.

Let $\epsilon > 0$ and choose $R > 0$ so that $|f(z)| < \epsilon$ for $|z| \geq R$.

Then $|f(z)| < \epsilon$ for $|z| = R$, the boundary of the disk $|z| \leq R$.

The maximum principle implies that $|f(z)| < \epsilon$ for $|z| < R$ as well.

It follows that $|f(z)| < \epsilon$ for all $z \in \mathbb{C}$.

As $\epsilon > 0$ was arbitrary, this can only mean that $f \equiv 0$.

This is the contradiction we sought. □