Morera's Theorem

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Complex Variables

Recall

A while ago we proved:

Theorem 1 (Cauchy's Theorem for a Disk)

Suppose $f(z)$ is analytic on an open disk D. Then:

1. *f has an antiderivative on F;*
2. $\displaystyle\int_\gamma f(z) = 0$ *for any loop γ in D.*

The main ingredient in our proof was:

Theorem 2 (Cauchy's Theorem for Rectangles)

Suppose $f(z)$ is analytic on a domain Ω. If $R \subset \Omega$ is a closed rectangular region, then

$$\int_{\partial R} f(z)\, dz = 0.$$

A close look at the proof of Cauchy's theorem for a disk shows that the hypothesis of analyticity was used only to establish that integrals around rectangles vanish.

Therefore, by the same proof, we can establish:

Lemma 1

Suppose f is continuous on an open disk D, and that $\int_{\partial R} f(z)\,dz = 0$ for every closed rectangular region R inside D. Then:

1. *f has an antiderivative on D;*
2. *$\displaystyle\int_\gamma f(z)\,dz = 0$ for all loops γ in D.*

Morera's Theorem

Because of the C^∞ nature of analytic functions, we have the following interesting consequence of Lemma 1.

Theorem 3 (Morera's Theorem)

Suppose f is continuous on a domain Ω, and that $\int_{\partial R} f(z)\,dz = 0$ for every closed rectangular region $R \subset \Omega$. Then f is analytic on Ω.

Proof. Let $z_0 \in \Omega$ and choose an open disk $D \subset \Omega$ containing z_0.

By Lemma 1, f has an antiderivative F on D. Hence F is analytic on D.

Therefore $F \in C^\infty(D)$. In particular $F' = f$ is differentiable on D.

So $f'(z_0)$ exists. Since $z_0 \in \Omega$ was arbitrary, it follows that f is analytic on Ω. □

Application

As an application, we use Morera's theorem to prove a rather general result on differentiation under the integral sign.

Theorem 4

Let $\Omega \subset \mathbb{C}$ be a domain and let $[a,b] \subset \mathbb{R}$ be a closed interval. Let $h : [a,b] \times \Omega \to \mathbb{C}$ be continuous. If $h(t,z)$ is analytic in z for each fixed $t \in [a,b]$, then

$$H(z) = \int_a^b h(t,z)\,dt$$

is analytic on Ω and

$$H'(z) = \frac{d}{dz}\int_a^b h(t,z)\,dt = \int_a^b \frac{\partial}{\partial z} h(t,z)\,dt.$$

Proof. We first claim that H is continuous.

To see this, fix $z_0 \in \Omega$ and let $D \subset \Omega$ be a closed disk around z_0.

The function h is continuous on $[a, b] \times D$, which is compact. Thus h is uniformly continuous there.

Let $\epsilon > 0$ and choose $\delta > 0$ so that $|h(t_1, z_1) - h(t_2, z_2)| < \epsilon$ whenever $|t_1 - t_2| + |z_1 - z_2| < \delta$.

Then, if $|z - z_0| < \delta$, we have $|h(t, z) - h(t, z_0)| < \epsilon$ for all $t \in [a, b]$.

Therefore

$$|H(z) - H(z_0)| = \left| \int_a^b h(t,z) - h(t,z_0)\, dt \right| \leq (b-a)\epsilon$$

for $|z - z_0| < \delta$, and H is continuous at z_0.

Now let R be a closed rectangle in Ω. By Cauchy's theorem, $\int_{\partial R} h(t,z)\, dz = 0$ for every $t \in [a,b]$.

Thus

$$0 = \int_a^b \int_{\partial R} h(t,z)\, dz\, dt = \int_{\partial R} \int_a^b h(t,z)\, dt\, dz = \int_{\partial R} H(z)\, dz.$$

Interchanging the order of integration can be justified by parametrizing the four sides of ∂R and appealing to Fubini's theorem.

Morera's theorem implies that H is analytic on Ω.

To compute H', fix $z_0 \in \Omega$ and choose $r > 0$ so that the circle $C_r = \{|z - z_0| = r\}$ is contained in Ω.

By the Cauchy integral formula (twice),

$$H'(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{H(z)}{(z - z_0)^2}\, dz = \frac{1}{2\pi i} \int_{C_r} \int_a^b \frac{h(t, z)}{(z - z_0)^2}\, dt\, dz$$

$$= \int_a^b \frac{1}{2\pi i} \int_{C_r} \frac{h(t, z)}{(z - z_0)^2}\, dz\, dt = \int_a^b \frac{\partial}{\partial z} h(t, z)\bigg|_{z = z_0} dt.$$

□

Example 1

Show that

$$H(z) = \int_0^1 e^{-z^2 t^2}\, dt$$

is entire, and compute $H'(z)$.

Solution. Let $h(t, z) = e^{-z^2 t^2}$.

Then h is clearly continuous on $[0, 1] \times \mathbb{C}$, and is entire in z for each fixed $t \in [0, 1]$.

By Theorem 4 H is entire, with

$$H'(z) = \int_0^1 \frac{\partial}{\partial z} e^{-z^2 t^2}\, dt = -2z \int_0^1 t^2 e^{-z^2 t^2}\, dt.$$

□