Infinite Series of Complex Numbers

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Complex Variables

Convergence of Series

An *(infinite) series* is an expression of the form

$$\sum_{k=1}^{\infty} a_k, \tag{1}$$

where $\{a_k\}$ is a sequence in $\mathbb{C}$.

We write $\sum a_k$ when the lower limit of summation is understood (or immaterial).

We call $S_n = \sum_{k=1}^{n} a_k$ the nth *partial sum* of (1).

We say that (1) *converges* to the *sum* $S = \lim_{n\to\infty} S_n$, when the limit exists. Otherwise (1) *diverges*.

The Cauchy Criterion for Series

The Cauchy criterion for the convergence of $\{S_n\}$ is that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that

$$|S_m - S_n| = \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon \quad \text{for all } N \le m < n.$$

Note that, in particular, $S_n - S_{n-1} = a_n$. The Cauchy criterion thus implies that $\lim_{n\to\infty} a_n = 0$. Hence:

Theorem 1 (Divergence Test)

If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$.

By considering partial sums, one finds that *convergent* series behave linearly, much as integrals do:

$$\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}(a_k + b_k), \qquad c\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(c \cdot a_k).$$

The product of two series is a *double series*:

$$\left(\sum_{k=1}^{\infty} a_k\right)\left(\sum_{\ell=1}^{\infty} b_\ell\right) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} a_k b_\ell. \tag{2}$$

If we group terms in (2) according to the sum of their indices, we formally obtain

$$\sum_{k=1}^{\infty}\left(\sum_{j+\ell=k} a_j b_\ell\right),$$

whose convergence and relation to (2) we consider below.

Absolute Convergence

A series $\sum a_k$ is called *absolutely convergent* provided $\sum |a_k|$ converges.

Note that there is no *a priori* connection between convergence and absolute convergence of a series.

Nonetheless, the two notions are connected in the "expected" way.

Theorem 2

If $\sum |a_k|$ converges, then $\sum a_k$ converges. Moreover,

$$\left|\sum a_k\right| \leq \sum |a_k|.$$

Remark. If $\sum a_k$ converges, but $\sum |a_k|$ diverges, we say that $\sum a_k$ is *conditionally convergent.*

Proof

Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ so that

$$\left|\sum_{k=m+1}^{n} a_k\right| \leq \sum_{k=m+1}^{n} |a_k| < \epsilon \quad \text{for all } N \leq m < n.$$

By the Cauchy criterion, this implies $\sum a_k$ converges.

Furthermore, for any n we have

$$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k| \leq \sum_{k=1}^{\infty} |a_k|,$$

because $|a_k| \geq 0$ for all k. The conclusion follows. $\square$

Cauchy Products of Series

Formal multiplication of series can also be justified with the additional hypothesis of absolute convergence.

Given series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, their *Cauchy product* is

$$\sum_{k=0}^{\infty} c_k, \quad \text{where} \quad c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

This is the natural way power series are multiplied, for example.

If $\sum a_k$ and $\sum b_k$ are conditionally convergent, their Cauchy product need not converge.

Convergence of Cauchy Products

However, we can show that:

Theorem 3

If $\sum_{k=0}^{\infty} a_k$ converges to A, $\sum_{k=0}^{\infty} b_k$ converges to B, and at least one of them converges absolutely, then their Cauchy product converges to AB.

Remark. If both series converge absolutely, it is not difficult to use Theorem 3 to show that their Cauchy product does, too.

Proof. Let $A_n = \sum_{k=0}^{n} a_k$, and likewise for B_n, C_n.

Assume (WLOG) that $\{A_n\}$ converges absolutely.

By reversing the order of summation, we find that

$$C_n = \sum_{k=0}^{n} c_k = \sum_{k=0}^{n} \sum_{j=0}^{k} a_j b_{k-j} = \sum_{j=0}^{n} a_j \sum_{k=j}^{n} b_{k-j}$$

$$= \sum_{j=0}^{n} a_j B_{n-j} = \sum_{j=0}^{n} a_{n-j} B_j = \sum_{j=0}^{n} a_{n-j}(B_j - B + B)$$

$$= \sum_{j=0}^{n} a_{n-j}(B_j - B) + A_n B.$$

Because $A_n \to A$, it therefore suffices to show that

$$\lim_{n\to\infty} \sum_{j=0}^{n} a_{n-j}(B_j - B) = 0.$$

Let $\epsilon > 0$ and choose $N_1 \in \mathbb{N}$ so that $|B_j - B| < \epsilon$ for $j \geq N_1$. Write

$$\sum_{j=0}^{n} a_{n-j}(B_j - B) = \underbrace{\sum_{j=0}^{N_1-1} a_{n-j}(B_j - B)}_{X} + \underbrace{\sum_{j=N_1}^{n} a_{n-j}(B_j - B)}_{Y}.$$

for $n \geq N_1$.

By assumption,

$$|Y| < \epsilon \sum_{j=N_1}^{n} |a_{n-j}| \leq \epsilon \sum_{k=0}^{\infty} |a_k| = \epsilon A',$$

by absolute convergence.

As $B_j - B \to 0$ as $j \to \infty$, there is an $M > 0$ so that $|B_j - B| \leq M$ for all j.

Thus
$$|X| \leq M \sum_{j=0}^{N_1-1} |a_{n-j}| = M \sum_{k=n-N_1+1}^{n} |a_k|.$$

Now choose $N_2 \in \mathbb{N}$ so that $\sum_{k=m+1}^{n} |a_k| < \epsilon$ for $n > m \geq N_2$.

Then for $n \geq N_1 + N_2$ we have $n > n - N_1 \geq N_2$, so that

$$|X| \leq M \sum_{k=n-N_1+1}^{n} |a_k| < M\epsilon,$$

and hence $|X + Y| \leq |X| + |Y| < (A' + M)\epsilon$. The result follows. $\square$

Geometric Series

A series of the form

$$\sum_{k=0}^{\infty} z^k, \tag{3}$$

with $z \in \mathbb{C}$ is called a *geometric series*.

The series (3) necessarily diverges if $|z| \geq 1$, since then

$$|z^k| = |z|^k \geq 1$$

for all k, and hence $z^k \not\to 0$.

For $|z| < 1$, the convergence of geometric series is governed by the *telescoping polynomial identity*

$$(X - 1)(X^n + X^{n-1} + \cdots + X + 1) = X^{n+1} - 1. \tag{4}$$

Indeed, if we evaluate (4) at $X = z$, then divide by $z - 1$, we obtain

$$\sum_{k=0}^{n} z^k = \frac{z^{n+1} - 1}{z - 1}. \tag{5}$$

Since $|z| < 1$, the RHS of (3) converges to $\frac{0-1}{z-1} = \frac{1}{1-z}$. Thus

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z} \quad \text{for } |z| < 1.$$

This, in turn, implies that

$$\sum_{k=0}^{\infty} |z^k| = \sum_{k=0}^{\infty} |z|^k = \frac{1}{1 - |z|} \quad \text{for } |z| < 1.$$

This proves:

Theorem 4 (Convergence of Geometric Series)

The geometric series $\sum z^k$ *converges absolutely for* $|z| < 1$ *and diverges otherwise. The sum is given by*

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \quad (|z| < 1).$$

Example 1

Show that $\displaystyle\sum_{k=1}^{\infty} kz^{k-1} = \frac{1}{(1-z)^2}$ for $|z| < 1$.

Solution. We square the geometric series and use the Cauchy product.

Because geometric series converge absolutely, Theorem 3 implies

$$\frac{1}{(1-z)^2} = \left(\sum_{k=0}^{\infty} z^k\right)^2 = \left(\sum_{k=0}^{\infty} z^k\right)\left(\sum_{k=0}^{\infty} z^k\right) = \sum_{k=0}^{\infty} c_k(z),$$

where

$$c_k(z) = \sum_{j=0}^{k} z^j z^{k-j} = \sum_{j=0}^{k} z^k = (k+1)z^k.$$

The result follows after reindexing the sum on the right. $\square$

Remark. It's worth noting that the identity

$$\sum_{k=1}^{\infty} k z^{k-1} = \frac{1}{(1-z)^2}$$

also results from formally differentiating the geometric series.

An Error Estimate

Just how rapidly does a geometric series converge?

If $|z| < 1$, we have

$$\left|\frac{1}{1-z} - \sum_{k=0}^{n} z^k\right| = \left|\sum_{k=n+1}^{\infty} z^k\right| \leq |z|^{n+1} \sum_{k=0}^{\infty} |z|^k = \frac{|z|^{n+1}}{1-|z|}.$$

So the partial sums converge exponentially (in n) to the infinite sum, more slowly as $|z| \to 1^-$.

Remarks

We have intentionally avoided discussing general convergence tests for series (e.g., the (limit) comparison tests, root test, ratio test, etc.).

These are (implicitly or explicitly) tests for absolute convergence.

As such they don't have true extensions to complex series, since they can simply be applied to $\sum |a_k|$, which is a *real* series.

An exception to this rule is *Dirichlet's test*, which generalizes the alternating series test.

Dirichlet's test for Convergence

Theorem 5 (Dirichlet's Test)

Let $\{a_k\}$ be a sequence of real numbers and let $\{b_k\}$ be a sequence of complex numbers. Suppose that:

1. $a_k \searrow 0$;
2. $\left|\sum_{k=1}^{N} b_k\right| \leq M$ *for all* $N \in \mathbb{N}$.

Then $\sum a_k b_k$ *converges.*

Remarks.

1. Dirichlet's test subsumes the alternating series (take $b_k = (-1)^k$).
2. The proof is a straightforward application of *partial summation*.

Proof

Let $B_n = \sum_{k=1}^n b_k$, with $B_0 = 0$. Then

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k (B_k - B_{k-1}) = \sum_{k=1}^{n} a_k B_k - \sum_{k=0}^{n-1} a_{k+1} B_k$$
$$= a_n B_n - a_1 B_0 - \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k$$
$$= a_n B_n - \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k.$$

This procedure is known as *partial summation*.

Since $|B_n| \leq M$ and $a_n \to 0$, $a_n B_n \to 0$.

The series $\sum(a_{k+1} - a_k)B_k$ converges absolutely since

$$\sum_{k=1}^{n} \left|(a_{k+1} - a_k)B_k\right| \leq M \sum_{k=1}^{n}(a_k - a_{k+1}) = M(a_1 - a_{n+1}),$$

and $a_{n+1} \to 0$. $\square$

Remark. As the proof shows, $\{a_n\}$ need not be real or decreasing, as long as $a_n \to 0$ and the series $\sum(a_{k+1} - a_k)$ converges absolutely.

Example

Example 2

Let $n \in \mathbb{N}$ and $\zeta = e^{2\pi i/n}$. Show that $\displaystyle\sum_{k=1}^{\infty} \frac{\zeta^k}{k^\sigma}$ converges for $\sigma > 0$.

Solution. Since ζ is a nontrivial nth root of unity, the sequence $b_k = \zeta^k$ is periodic and satisfies

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0.$$

It follows that the partial sums $\sum_{k \le N} b_k$ are also periodic, and are therefore bounded.

Since $a_k = \frac{1}{k^\sigma} \searrow 0$, the result follows from Dirichlet's test. $\square$