Taylor Series

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Complex Variables

Introduction

We will be motivated by the following **Question**:

Which analytic functions are given by power series?

We begin with an observation.

Suppose $f(z)$ is given by a power series $\sum a_k z^k$ with radius of convergence $R > 0$.

According to results of last time, we can formally differentiate the PS expression for f arbitrarily often without changing the radius:

$$f^{(m)}(z) = \frac{d^m}{dz^m} \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} \frac{d^m}{dz^m} a_k z^k = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m}$$

Writing out the first few terms this becomes

$$f^{(m)}(z) = m!a_m + (m+1)!a_{m+1}z + \frac{(m+2)!}{2!}a_{m+2}z^2 + \cdots$$

Note that this implies

$$f^{(m)}(0) = m!a_m \iff a_m = \frac{f^{(m)}(0)}{m!}. \tag{1}$$

Moral. If a function $f(z)$ is given by a PS centered at $z_0 = 0$, the coefficients are completely determined by the values of f and its derivatives at z_0. That is:

Theorem 1 (Identity Principle)

If $\sum a_k z^k = \sum b_k z^k$ in a neighborhood of $z_0 = 0$, then $a_k = b_k$ for all k.

Taylor Series

So if we are given $f(z)$ and a central point z_0, there is *only one possible* PS expansion for $f(z)$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

This is the *Taylor series* of f centered at z_0.

Our **Question** then becomes: which analytic $f(z)$ are equal to their Taylor series?

As we shall see, the somewhat amazing answer is: *all of them*!

Deriving Taylor Series Expansions

In the realm of real variables, one derives Taylor expansions via integration by parts.

We can avoid this approach altogether by taking advantage of the Cauchy integral formula.

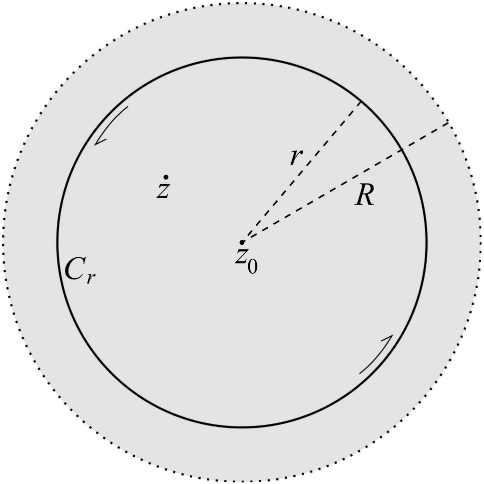
The Set-up. Suppose f is analytic at z_0.

Then there is an $R > 0$ so that f is analytic on the disk $|\zeta - z_0| < R$.

Fix z inside the disk as well as r satisfying $|z - z_0| < r < R$.

Let C_r denote the circle $|\zeta - z_0| = r$.

The Set-Up



According to the Cauchy integral formula,

$$
\begin{aligned}
f(z) = \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta - z}\,d\zeta &= \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta - z_0 + z_0 - z}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta - z_0}\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{C_r}\frac{f(\zeta)}{\zeta - z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^k\,d\zeta,
\end{aligned}
$$

since $\left|\frac{z-z_0}{\zeta-z_0}\right| = \frac{|z-z_0|}{r} < \frac{r}{r} = 1$ on C_r.

This also shows that the series in the integral is bounded term-wise by the convergent geometric series with ratio $|z - z_0|/r < 1$.

The M-test implies the series therefore converges uniformly in ζ.

Thus we can interchange the order of integration and summation:

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - z_0} \sum_{k=0}^{\infty} \left(\frac{z-z_0}{\zeta - z_0}\right)^k \, d\zeta \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta\right)(z-z_0)^k \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k, \qquad (2)
\end{aligned}
$$

where we have applied the Cauchy integral formula for derivatives.

Conclusion

The Taylor expansion (2) is valid for any z within the disk $|z - z_0| < R$ where f is analytic.

We have therefore proven:

Theorem 2

Let $f(z)$ be analytic at z_0. Then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \quad (3)$$

for all z in the largest disk centered at z_0 on which f is analytic. The radius of this disk is the radius of convergence of the Taylor series (3).

Examples

Example 1

Prove that $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ for all $z \in \mathbb{C}$.

Solution. The function $f(z) = e^z$ is entire.

It will therefore agree with its Taylor series expansion about any point.

Since $f^{(k)}(z) = e^z$ for all k, we have

$$e^z = \sum_{k=0}^{\infty} \frac{e^0}{k!} z^k = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

for all $z \in \mathbb{C}$. □

Example 2

Compute the Taylor expansion of $\sin z$ about $z_0 = 0$.

Solution. For any $z \in \mathbb{C}$ we have

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}\left(\sum_{k=0}^{\infty} \frac{(iz)^k}{k!} - \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!}\right)$$

$$= \frac{1}{2i} \sum_{k=0}^{\infty} i^k \underbrace{\left(1 + (-1)^{k+1}\right)}_{=0 \text{ for } k=2\ell} \frac{z^k}{k!} = \frac{1}{i} \sum_{\ell=0}^{\infty} i^{2\ell+1} \frac{z^{2\ell+1}}{(2\ell+1)!}$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell z^{2\ell+1}}{(2\ell+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

By the Identity Principle, this must be the Taylor expansion. $\square$

Example 3

Compute the Taylor expansion of $\cos z$ about $z_0 = 0$.

Solution. Recall that we are permitted to formally derive power series.

Therefore, for all $z \in \mathbb{C}$ we have

$$\cos z = \frac{d}{dz}\sin z = \frac{d}{dz}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}z^{2\ell+1}}{(2\ell+1)!} = \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(2\ell+1)z^{2\ell}}{(2\ell+1)!}$$
$$= \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}z^{2\ell}}{(2\ell)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

which is the Taylor expansion by the Identity Principle. □

Another Example

We showed that $\operatorname{Arctan} z = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell z^{2\ell+1}}{2\ell+1}$ for $|z| < 1$.

What is limiting the radius of convergence?

We recall that $\operatorname{Arctan} z = \frac{1}{2i} \operatorname{Log}\left(\frac{1+iz}{1-iz}\right)$.

The logarithm has a branch cut along $(-\infty, 0]$, where it fails to be analytic.

The preimage of the cut under the FLT $z \mapsto \frac{1+iz}{1-iz}$ is

$$C = (-i\infty, -i] \cup [i, i\infty).$$

This is where $\operatorname{Arctan} z$ fails to be analytic.

The largest disk centered at $z_0 = 0$ contained in $\mathbb{C} \setminus C$ has radius $R = 1$.

Therefore the radius of convergence of the Taylor expansion is precisely $R = 1$.