Zeros of Analytic Functions

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Complex Variables

Zeros

As with polynomials, analytic functions can have "repeated roots," and these are detected using derivatives.

Let f be analytic at z_0. We say that f has a *zero at* z_0 if $f(z_0) = 0$.

We say a zero z_0 has *order* $m \in \mathbb{N}$ if

$$f^{(k)}(z_0) = 0 \text{ for } 0 \leq k \leq m - 1, \text{ and } f^{(m)}(z_0) \neq 0.$$

A zero of order 1 is called *simple*. Zeros of higher order are called *double*, *triple*, etc.

We say a zero z_0 has *infinite order* if

$$f^{(k)}(z_0) = 0 \text{ for all } k \geq 0.$$

Examples

Example 1

Show that the zeros of $f(z) = \sin z$ are all simple.

Solution. The zeros of $\sin z$ are $z_0 = n\pi$, $n \in \mathbb{Z}$.

Since $f'(z) = \cos z$ and $\cos n\pi = \pm 1 \neq 0$, every zero is simple. □

Example 2

Show that $f(z) = \cos z - e^z + z$ has a double zero at $z_0 = 0$.

Solution. We have $f(0) = \cos 0 - e^0 + 0 = 0$,
$f'(0) = -\sin 0 - e^0 + 1 = 0$ and $f''(0) = -\cos 0 - e^0 = -2$.

Since $-2 \neq 0$, the result follows. □

Zeros of Infinite Order

Our first task is to show that zeros of infinite order aren't very interesting.

Theorem 1

Let f be analytic on a domain Ω. If f has a zero of infinite order at $z_0 \in \Omega$, then $f \equiv 0$.

Proof. Let $Z = \{z_0 \in \Omega \,|\, f \text{ has an infinite order zero at } z_0\}$.

We will show that Z is clopen in Ω. Consequently, if $Z \neq \varnothing$, then $Z = \Omega$, and the result follows.

Let $Z_k = \{z_0 \in \Omega \,|\, f^{(k)}(z_0) = 0\}$. Then Z_k is closed and so

$$Z = \bigcap_{k \in \mathbb{N}_0} Z_k \quad \text{is closed.}$$

Let $z_0 \in Z$. We know that f agrees with its Taylor series on an open disk D centered at z_0:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=0}^{\infty} \frac{0}{k!}(z - z_0)^k = 0 \quad \text{on} \quad D.$$

Since $f \equiv 0$ on D, $D \subset Z$.

This proves Z is open, and completes the proof. □

Corollary 1

A nonconstant analytic function on a domain Ω only has zeros of finite order in Ω.

Zeros of Finite Order

Suppose f is analytic on a domain Ω with a zero of finite order m at $z_0 \in \Omega$. Then on an open disk $D \subset \Omega$ centered at z_0 one has

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = \sum_{k=m}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \\ &= (z-z_0)^m \sum_{k=m}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-m} \\ &= (z-z_0)^m \underbrace{\left(b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots\right)}_{g(z)}, \end{aligned}$$

where $g(z_0) = b_0 = f^{(m)}(z_0)/m! \neq 0$. The function $g(z)$ is analytic on D (it's given by a convergent PS).

The function $f(z)/(z-z_0)^m$ is analytic on $\Omega \setminus \{z_0\}$, and agrees with g on $D \setminus \{z_0\}$.

It follows that g extends to be analytic throughout Ω, and satisfies $f(z) = (z-z_0)^m g(z)$ there. All together this proves half of:

Theorem 2

Let f be analytic on a domain Ω. Then f has a zero of order m at $z_0 \in \Omega$ if and only if $f(z) = (z-z_0)^m g(z)$, where g is analytic on Ω and satisfies $g(z_0) \neq 0$.

Proof. The converse direction follows from direct computation (induction), or from substitution of the Taylor expansion of g at z_0. □

Remarks

1. Theorem 2 says that we can "factor out" the zeros of an analytic function in the same way we can with polynomials.
2. Theorem 2 also says that if $f(z)$ has an order m zero at z_0, then $g(z) = f(z)/(z - z_0)^m$ can be analytically continued to z_0, i.e. the singularity at z_0 is *removable*.

Example 3

Because $f(z) = \sin z$ has a simple zero at $z_0 = 0$, the function $\operatorname{sinc} z = \frac{\sin z}{z}$ can be analytically continued to $z_0 = 0$. In fact

$$\operatorname{sinc} z = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$$

for all $z \in \mathbb{C}$. In particular, $\operatorname{sinc}(0) = 1$.

Isolation of Zeros

An immediate consequence of the factorization property of zeros is:

Corollary 2

Let f be a nonconstant analytic on a domain Ω. Then the zeros of f in Ω are isolated.

Proof. Suppose $z_0 \in \Omega$ is a zero of f.

Because f is nonconstant, f has finite order m at z_0.

Write $f(z) = (z - z_0)^m g(z)$ with g analytic on Ω and $g(z_0) \neq 0$.

Because g is continuous and nonzero at z_0, there is an open disk D at z_0 for which $g(z) \neq 0$ for all $z \in D$.

Then f vanishes *only* at z_0 in D. So z_0 is isolated by D. □

Identity Principle

Another immediate consequence is the:

Theorem 3 (Identity Principle)

Let f and g be analytic on a domain Ω and let $E \subset \Omega$. If $f|_E = g|_E$ and E contains a limit point $z_0 \in E$, then $f \equiv g$.

Proof. The function $h = f - g$ is analytic on Ω, vanishes on E, and has a nonisolated zero at z_0.

Thus $h \equiv 0$ by (the contrapositive of) Corollary 2, and hence $f \equiv g$. □

Example

Example 4

Prove that $f(x+iy) = e^x(\cos y + i\sin y)$ is the only possible analytic extension of e^x from $\mathbb{R}$ to $\mathbb{C}$.

Solution. We know that $f(z) = e^z$ is entire and satisfies $f(x) = e^x$ for $x \in \mathbb{R}$.

Suppose that $g(z)$ is also entire and satisfies $g(x) = e^x$ for $x \in \mathbb{R}$.

Then $g(x) = f(x)$ for all $x \in \mathbb{R}$.

Since $\mathbb{R}$ has limit points, the Identity Principle implies $g \equiv f$. □

Example

One can easily verify that $\overline{e^z} = e^{\overline{z}}$. This is a consequence of the following general principle.

Theorem 4 (Reflection Principle)

Suppose f is entire and $f(\mathbb{R}) \subset \mathbb{R}$. Then $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$.

Proof. We have seen that $g(z) = \overline{f(\overline{z})}$ is analytic if $f(z)$ is.

If $x \in \mathbb{R}$, then

$$g(x) - f(x) = \overline{f(\overline{x})} - f(x) = \overline{f(x)} - f(x) = f(x) - f(x) = 0,$$

since $f(x)$ is real.

So $g|_{\mathbb{R}} = f|_{\mathbb{R}}$, and hence $g \equiv f$ by the Identity Principle. □

Analytic Continuation

Definition

Let $\Omega \subset \Omega'$ be domains. Suppose f is analytic on Ω. We say that $\widehat{f}$ is an *analytic continuation* of f (to Ω') provided $\widehat{f}$ is analytic on Ω' and satisfies $\widehat{f}|_\Omega \equiv f$.

The remarkable fact about analytic continuations is that, when they exist, they are unique.

Theorem 5 (Uniqueness of Analytic Continuation)

Let $\Omega \subset \Omega'$ be domains and suppose f is analytic on Ω. If f has an analytic continuation to Ω', then it is unique.

Proof. Suppose $\widehat{f} : \Omega' \to \mathbb{C}$ and $\widetilde{f} : \Omega' \to \mathbb{C}$ are both analytic continuations of f to Ω'.

Then $\widehat{f}|_\Omega \equiv f \equiv \widetilde{f}|_\Omega$.

Because Ω is nonempty and open, it has limit points.

The Identity Principle implies $\widehat{f} \equiv \widetilde{f}$. □

The formulae defining analytic functions frequently have natural limitations on their domains, despite the fact that the functions themselves may have larger domains.

Power series are an example of this phenomenon.

Uniqueness of Analytic Continuation ensures that no matter how we choose to extend a given formula for an analytic function, the resulting extension will always be the same.

The Riemann Zeta Function

The *Riemann zeta function* is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{1}$$

Because

$$n^z = e^{z \ln n} = e^{x \ln n} e^{iy \ln n} = n^x e^{iy \ln n} \quad \Rightarrow \quad |n^z| = n^x,$$

for $x \geq x_0 > 1$ one has

$$\left| \frac{1}{n^z} \right| = \frac{1}{n^x} \leq \frac{1}{n^{x_0}}.$$

Since $\sum 1/n^{x_0}$ converges, the M-test implies (1) converges absolutely and uniformly on $x \geq x_0$.

It follows that (1) converges absolutely and normally on $x > 1$.

Therefore $\zeta(z)$ is an analytic function on $x > 1$.

We will extend $\zeta(z)$ to $\{x > 0\} \setminus \{1\}$. For any $N \in \mathbb{N}$ consider the partial sum

$$\sum_{n=1}^{N} \frac{1}{n^z} = \sum_{n=1}^{N} \left(n - (n-1)\right) n^{-z} = \sum_{n=1}^{N} n \cdot n^{-z} - \sum_{n=1}^{N} (n-1) n^{-z}$$

$$= \sum_{n=1}^{N} n \cdot n^{-z} - \sum_{n=0}^{N-1} n(n+1)^{-z}$$

$$= N^{1-z} - \sum_{n=1}^{N-1} n \left((n+1)^{-z} - n^{-z}\right)$$

$$= N^{1-z} - \sum_{n=1}^{N-1} n \int_n^{n+1} -zt^{-z-1}\,dt = N^{1-z} + z \sum_{n=1}^{N-1} \int_n^{n+1} [t] t^{-z-1}\,dt$$

$$= N^{1-z} + z \int_1^N [t] t^{-z-1}\,dt = N^{1-z} + z \int_1^N (\underbrace{[t] - t}_{\{t\}} + t) t^{-z-1}\,dt$$

$$= N^{1-z} - z \int_1^N \{t\} t^{-z-1}\,dt + z \int_1^N t^{-z}\,dt$$

$$= N^{1-z} - z \int_1^N \{t\} t^{-z-1}\,dt + z \frac{t^{1-z}}{1-z}\bigg|_1^N$$

$$= N^{1-z}\left(1 + \frac{z}{1-z}\right) + \frac{z}{z-1} - z \int_1^N \{t\} t^{-z-1}\,dt.$$

Since $|N^{1-z}| = N^{1-x}$, for $x > 1$, $N^{1-z} \to 0$ as $N \to \infty$.

Letting $N \to \infty$ above we therefore obtain

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \frac{z}{z-1} - z \int_1^{\infty} \{t\} t^{-z-1}\, dt, \tag{2}$$

for $x > 1$. Because $0 \le \{t\} < 1$ and $|t^{-z-1}| = t^{-x-1}$, the integral actually converges (absolutely) for $x > 0$!

Since $h(t, z) = \{t\} t^{-z-1}$ is analytic in z, we can use Morera's and Fubini's theorems to show the integral is analytic in z for $x > 0$.

Thererfore (2) provides the (unique) analytic continuation of $\zeta(z)$ to $\{x > 0\} \setminus \{1\}$!