BASIC COMPLEX ANALYSIS

Third Edition

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9. Evaluate $\int_\gamma \sqrt{z}\,dz$, where γ is the upper half of the unit circle: first, directly, and second, using the Fundamental Theorem 2.1.7.

10.• Evaluate $\int_\gamma \sqrt{z^2-1}\,dz$, where γ is a circle of radius $\frac{1}{2}$ centered at 0.

11. Evaluate

$$\int_\gamma \frac{2z^2-15z+30}{z^3-10z^2+32z-32}dz,$$

where γ is the circle $|z| = 3$. *Hint*: Use partial fractions; one root of the denominator is $z = 2$.

2.3 A Closer Look at Cauchy's Theorem

In this section we take another look at some of the issues that were treated informally in the previous section. The strategy is to start by carefully examining Cauchy's Theorem for a rectangle and then to use the theorem in this special case, together with subdivision arguments, to build up to more general regions in a systematic way.

Recall that the basic theme of Cauchy's Theorem is that *if a function is analytic everywhere inside a closed contour, then its integral around that contour must be* 0. The principal goal of this section is to give a proof of a form of the theorem known as a *homotopy version of Cauchy's Theorem*. This approach extends and sharpens the idea presented in the preceding section of the continuous deformation of a curve. The primary objective will be the precise formulation and proof of deformation theorems which say, roughly, that if a curve is continuously deformed through a region in which a function is analytic, then the integral along the curve does not change. The reader will also notice that in this section references are made not to "simple closed curves" but only to "closed curves."

Cauchy's Theorem for a Rectangle We begin with a careful statement of Cauchy's Theorem in this case.

Theorem 2.3.1 (Cauchy's Theorem for a Rectangle) *Suppose R is a rectangular path with sides parallel to the axes and that f is a function defined and analytic on an open set G containing R and its interior. Then $\int_R f = 0$.*

There are several methods to prove Cauchy's Theorem for a rectangle. One way, which fits the spirit of the previous section, is to prove a strong version of Green's Theorem for rectangles[4]. Another technique, the one that we follow, is a bisection technique due to Édouard Goursat in 1884. It was Goursat[5] who first noticed that

[4] F. Acker, The missing link, *Mathematical Intelligencer*, **18** (1996), 4–9.

[5] *Acta Mathematica*, **4** (1884), 197–200 and *Transactions of the American Mathematical Society*, **1** (1900), 14–16.

one does not need to assume that the derivative of f is continuous. Surprisingly, this follows automatically, which is a rather different situation than that for real functions of several variables.

Besides these techniques, there have been many other proofs of Cauchy's Theorem. For example, Pringsheim[6] uses triangles rather than rectangles, which has some advantages. Cauchy's original proof (for which the assumptions of continuity of the derivative were not made clear), had the content of Green's Theorem implicit in the argument—in fact Green did not formulate Green's Theorem as such until about 1830, whereas Cauchy presented his theorem in 1825.[7] There are also interesting proofs based on "homology" given by Ahlfors.[8]

Local Version of Cauchy's Theorem Before proving Cauchy's Theorem for a rectangle, we indicate how it can already be used to prove a limited but still important and more general case of Cauchy's Theorem.

Theorem 2.3.2 (Cauchy's Theorem for a Disk) *Suppose that $f : D \to \mathbb{C}$ is analytic on a disk $D = D(z_0; \rho) \subset \mathbb{C}$. Then*

(i) *f has an antiderivative on D; that is, there is a function $F : D \to \mathbb{C}$ that is analytic on D and that satisfies $F'(z) = f(z)$ for all z in D.*

(ii) *If Γ is any closed curve in D, then $\int_\Gamma f = 0$.*

From the discussion in §2.1 on the path independence of integrals (see Theorem 2.1.9), we know that (i) and (ii) are equivalent in the sense that whichever we establish first, the other will follow readily from it. Our problem is how to obtain either one of them. In the proof of the Path Independence Theorem 2.1.9, it was shown that (ii) follows easily from (i), and the construction of an antiderivative to get (i) was facilitated by the path independence of integrals. The strategy for proceeding is quite interesting.

1. Prove (ii) directly for the very special case in which Γ is the boundary of a rectangle.

2. Show that this limited version of path independence is enough to carry out a construction of an antiderivative similar to that in the proof of the Path Independence Theorem.

3. With (i) thus established, part (ii) in its full generality follows as in the Path Independence Theorem.

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[6] *Transactions of the American Mathematical Society*, **2** (1902)

[7] In his *Mémoire sur les intégrales définies prises entre des limites imaginaires.*

[8] L. Ahlfors, *Complex Analysis*, Second Edition (New York: McGraw-Hill, 1966).

Proof of Cauchy's Theorem for a Rectangle A subtle technical point worth repeating: Care must be taken because we do not know in advance that the derivative of f is continuous. In fact, we will use Cauchy's Theorem itself to eventually prove that f' is *automatically continuous.* Now let's get down to the proof.

Let P be the perimeter of R and Δ the length of its diagonal. Divide the rectangle R into four congruent smaller rectangles $R^{(1)}, R^{(2)}, R^{(3)}$, and $R^{(4)}$. If each is oriented in the counterclockwise direction, then cancellation along common edges leaves

$$\int_R f = \int_{R^{(1)}} f + \int_{R^{(2)}} f + \int_{R^{(3)}} f + \int_{R^{(4)}} f.$$

Since

$$\left|\int_R f\right| \le \left|\int_{R^{(1)}} f\right| + \left|\int_{R^{(2)}} f\right| + \left|\int_{R^{(3)}} f\right| + \left|\int_{R^{(4)}} f\right|,$$

there must be at least one of the rectangles for which $|\int_{R^{(k)}} f| \ge \frac{1}{4}|\int_R f|$. Call this subrectangle R_1. Notice that the perimeter and diagonal of R_1 are half those of R (Figure 2.3.1).

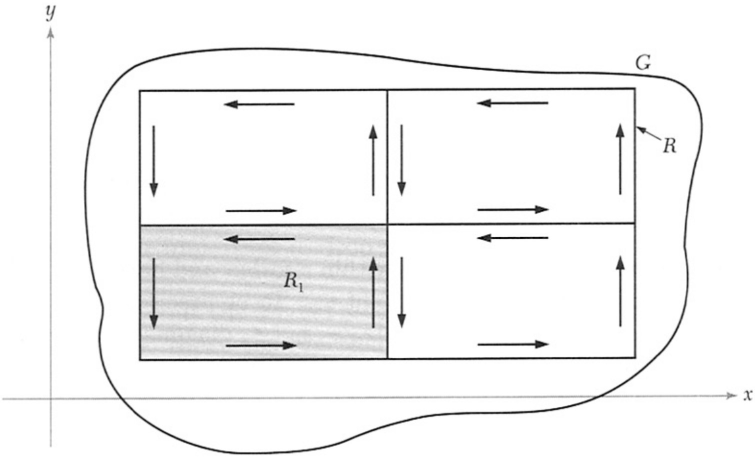

Figure 2.3.1: Bisection procedure.

Now repeat this bisection process, obtaining a sequence $R_1, R_2, R_3, \ldots$ of smaller and smaller rectangles that have the following properties:

(i) $\left|\int_{R_n} f\right| \ge \frac{1}{4}\left|\int_{R_{n-1}} f\right| \ge \cdots \ge \frac{1}{4^n}\left|\int_R f\right|$

(ii) Perimeter$(R_n) = \dfrac{1}{2^n}$ perimeter$(R) = \dfrac{P}{2^n}$

(iii) $\text{Diagonal}(R_n) = \dfrac{1}{2^n}\,\text{diagonal}(R) = \dfrac{\Delta}{2^n}$ (see Figure 2.3.2)

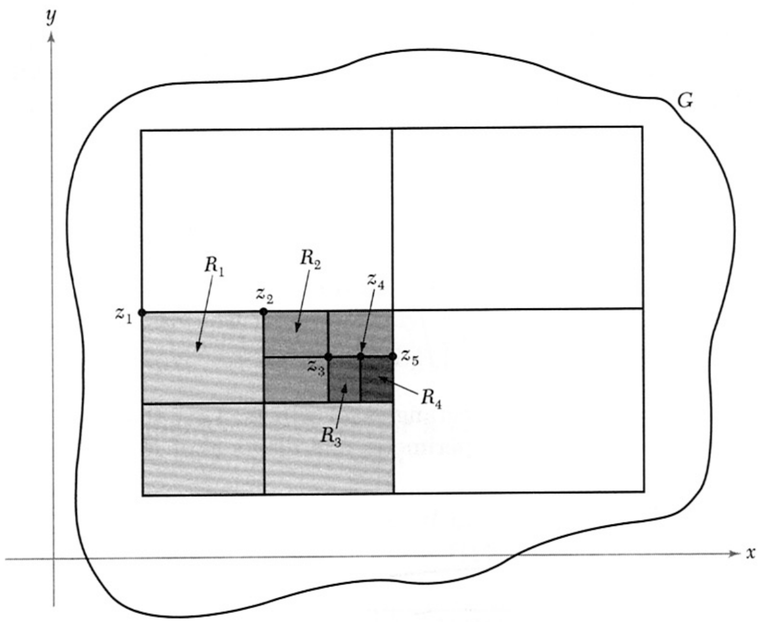

Figure 2.3.2: Goursat's repeated bisection process for the proof of Cauchy's Theorem for a rectangle.

Since these rectangles are nested one within another and have diagonals tending to 0, they must shrink down to a single point w_0. To be precise, let z_n be the upper left-hand corner of R_n. If $m > n$, then $|z_n - z_m| \le$ diagonal $(R_n) = \Delta/2^n$, and thus $\{z_n\}$ forms a Cauchy sequence that must converge to some point w_0. If z is any point on the rectangle R_n, then since all z_k with $k \ge n$ are within R_n, z can be no farther from w_0 than the length of the diagonal of R_n. That is, $|z - w_0| \le \Delta/2^n$ for z in R_n.

From (i) we see that $|\int_R f| \le 4^n |\int_{R_n} f|$. To obtain a sufficiently good estimate on the right side of this inequality, we use the differentiability of f at the point w_0.

For $\epsilon > 0$, there is a number $\delta > 0$ such that

$$\left| \frac{f(z) - f(w_0)}{z - w_0} - f'(w_0) \right| < \epsilon$$

whenever $|z - w_0| < \delta$. If we choose n large enough that $\Delta/2^n$ is less than δ, then

$$|f(z) - f(w_0) - (z - w_0) f'(w_0)| < \epsilon |z - w_0| \le \epsilon \frac{\Delta}{2^n}$$

for all points z on the rectangle R_n. Furthermore, by the Path Independence Theorem 2.1.9,

$$\int_{R_n} 1 dz = 0 \quad \text{and} \quad \int_{R_n} (z - w_0) dz = 0.$$

Since z is an antiderivative for $1, (z - w_0)^2/2$ is an antiderivative for $(z - w_0)$, and the path R_n is closed. Thus,

$$\begin{aligned}
\left|\int_R f\right| &\leq 4^n \left|\int_{R_n} f\right| \\
&= 4^n \left|\int_{R_n} f(z)dz - f(w_0)\int_{R_n} 1dz - f'(w_0)\int_{R_n}(z - w_0)dz\right| \\
&\leq 4^n \left|\int_{R_n} [f(z) - f(w_0) - (z - w_0)f'(w_0)]dz\right| \\
&\leq 4^n \int_{R_n} |f(z) - f(w_0) - (z - w_0)f'(w_0)|\,|dz| \\
&\leq 4^n \left(\frac{\epsilon\Delta}{2^n}\right) \cdot \text{perimeter } (R_n) \\
&\leq \epsilon\Delta P.
\end{aligned}$$

Since this is true for every $\epsilon > 0$, we must have $|\int_R f| = 0$ and so $\int_R f = 0$, as desired. ■

Back to Cauchy's Theorem on a Disk For most of the rest of this section, "curve" means "piecewise C^1 curve." However, at one point in the technical development it will become important to drop this piecewise C^1 restriction and consider continuous curves.[9]

We can now carry out the second step of the proof of Cauchy's Theorem for a disk (Theorem 2.3.2). Since the function f is analytic on the disk $D = D(z_0; \rho)$, the result for a rectangle just proved shows that the integral of f is 0 around any rectangle in D. This is enough to carry out a construction of an antiderivative for f very much like that done in the proof of the Path Independence Theorem 2.1.9 and thus to establish part (i) of the theorem.

We will again define the antiderivative $F(z)$ as an integral from z to z_0. However, we do not yet know that such an integral is path independent. Instead we will specify a particular choice of path and use the new information available—the analyticity of f and the geometry of the situation together with the rectangular case of Cauchy's Theorem—to show that we get an antiderivative. For the duration of this proof we will use the notation $\langle\langle a, b\rangle\rangle$ to denote the polygonal path proceeding from a point a to a point b in two segments, first parallel to the x axis, then parallel to the y axis, as in Figure 2.3.3.

[9]The technical treatment of integration over continuous curves is given in the Internet Supplement.

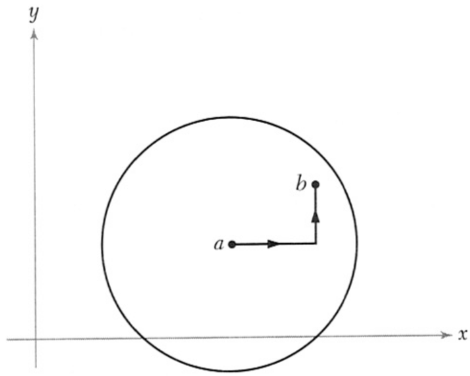

Figure 2.3.3: The path $\langle\langle a, b\rangle\rangle$.

If the point b is in a disk $D(a;\delta)$ centered at a, then the path $\langle\langle a, b\rangle\rangle$ is contained in that disk. Thus, for $z \in D$, we may define a function $F(z)$ by

$$F(z) = \int_{\langle\langle z_0, z\rangle\rangle} f(\xi)d\xi.$$

We want to show that $F'(z) = f(z)$. To do this we need to show that

$$\lim_{w\to z} = \frac{F(w) - F(z)}{w - z} = f(z).$$

Fixing $z \in D$ and $\epsilon > 0$, we use the fact that D is open and f is continuous on D to choose $\delta > 0$ small enough that $D(z;\delta) \subset D$ and $|f(z) - f(\xi)| < \epsilon$ for $\xi \in D(z;\delta)$. If $w \in D(z;\delta)$, then the path $\langle\langle z, w\rangle\rangle$ is contained in $D(z;\delta)$ and hence in D. The paths $\langle\langle z_0, z\rangle\rangle$ and $\langle\langle z_0, w\rangle\rangle$ are also contained in D, and these three paths fit together in a nice way with a rectangular path R also contained in D and having one corner at z; see Figure 2.3.4. We can write, for the two cases in Figure 2.3.4,

$$\int_{\langle\langle z_0, z\rangle\rangle} f(\xi)d\xi \pm \int_R f(\xi)d\xi + \int_{\langle\langle z, w\rangle\rangle} f(\xi)d\xi = \int_{\langle\langle z_0, w\rangle\rangle} f(\xi)d\xi.$$

By the Cauchy theorem for a rectangle, $\int_R f(\xi)d\xi = 0$, so the preceding equation becomes

$$F(z) + \int_{\langle\langle z, w\rangle\rangle} f(\xi)d\xi = F(w).$$

Neither side of the right triangle defined by $\langle\langle z, w\rangle\rangle$ can be any longer than its

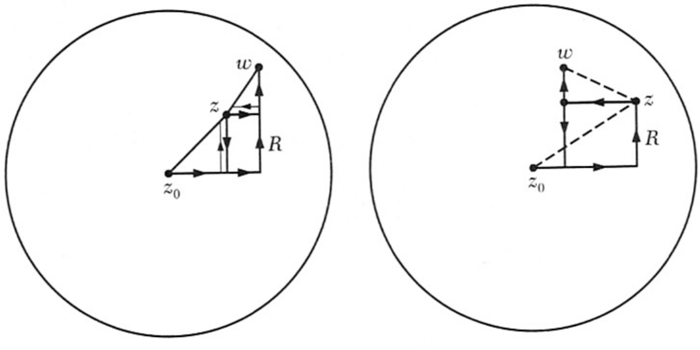

Figure 2.3.4: Two possible configurations for R, z_0, z, and w.

hypotenuse, which has length $|z-w|$, so length$(\langle\langle z,w\rangle\rangle) \leq 2|z-w|$, and thus

$$\begin{aligned}
\left|\frac{F(w)-F(z)}{w-z} - f(z)\right| &= \frac{1}{|w-z|}\left|\int_{\langle\langle z,w\rangle\rangle} f(\xi)d\xi - f(z)(w-z)\right| \\
&= \frac{1}{|w-z|}\left|\int_{\langle\langle z,w\rangle\rangle} f(\xi)d\xi - f(z)\int_{\langle\langle z,w\rangle\rangle} 1d\xi\right| \\
&= \frac{1}{|w-z|}\left|\int_{\langle\langle z,w\rangle\rangle} [f(\xi)-f(z)]d\xi\right| \\
&\leq \frac{1}{|w-z|}\int_{\langle\langle z,w\rangle\rangle} |f(\xi)-f(z)||d\xi| \\
&\leq \frac{1}{|w-z|}\epsilon \text{ length}(\langle\langle z,w\rangle\rangle) \leq \frac{1}{|w-z|}\epsilon\cdot 2|w-z| = 2\epsilon.
\end{aligned}$$

Thus,

$$\lim_{w\to z} = \frac{F(w)-F(z)}{w-z} = f(z)$$

and therefore $F'(z) = f(z)$, as desired. This establishes part (i) of the theorem. Since f has an antiderivative defined everywhere on D and γ is a closed curve in D, we have $\int_\gamma f = 0$ by the Path Independence Theorem 2.1.9. This establishes part (ii) of the theorem and so the proof of Cauchy's Theorem in a disk (Theorem 2.3.2) is now complete. ■

Deleted Neighborhoods For technical reasons that will be apparent in §2.4, it will be useful to have the following variant of Cauchy's Theorem for a rectangle.

Lemma 2.3.3 *Suppose that R is a rectangular path with sides parallel to the axes, that f is a function defined on an open set G containing R and its interior, and*