Complex nth Roots

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1 Roots of Unity

For $n \in \mathbb{N}$, the complex solutions of the equation $z^n = 1$ are called the *nth roots of unity*. We let μ_n denote the set of nth roots of unity in $\mathbb{C}$. Note that we always have $1 \in \mu_n$ so that $|\mu_n| \geq 1$. On the other hand, because the nth roots of unity are the roots of the degree n polynomial $X^n - 1$, and $\mathbb{C}$ is a field, $|\mu_n| \leq n$. We will prove that, in fact, $|\mu_n| = n$. That is, $z^n = 1$ has exactly n solutions in $\mathbb{C}$.

We begin with some convenient notation. For $z_1, z_2, a \in \mathbb{C}$, we say z_1 *is congruent to* z_2 *modulo* a, and write $z_1 \equiv z_2 \pmod{a}$, whenever $z_1 - z_2 \in a\mathbb{Z}$, or equivalently when $z_1 = z_2 + na$ for some $n \in \mathbb{Z}$. It is easy to check that congruence modulo a is an equivalence relation, and that if $z_1 \equiv z_2 \pmod{a}$, then $wz_1 \equiv wz_2 \pmod{wa}$ for all $w \in \mathbb{C}$.[1]

The complex numbers of modulus 1 are those with polar representation $e^{i\theta}$, $\theta \in \mathbb{R}$. Because θ represents the argument of $e^{i\theta}$, we immediately conclude that $e^{i\theta_1} = e^{i\theta_2}$ if and only if θ_1 and θ_2 differ by a multiple of 2π, i.e. $\theta_1 \equiv \theta_2 \pmod{2\pi}$. In particular, $e^{i\theta} = 1 = e^{i0}$ if and only if $\theta \equiv 0 \pmod{2\pi}$.

Now let's compute μ_n. Write $z = re^{i\theta}$ with $r > 0$ and $\theta \in \mathbb{R}$. Then

$$z^n = 1 \;\Leftrightarrow\; r^n e^{in\theta} = 1 \;\Leftrightarrow\; \left\{\begin{matrix} r^n = 1 \\ \text{and} \\ n\theta \equiv 0 \pmod{2\pi} \end{matrix}\right\} \;\Leftrightarrow\; \left\{\begin{matrix} r = 1 \\ \text{and} \\ \theta \equiv 0 \pmod{\frac{2\pi}{n}} \end{matrix}\right\} \;\Leftrightarrow\; \left\{\begin{matrix} r = 1 \\ \text{and} \\ \theta = \frac{2\pi j}{n}, j \in \mathbb{Z} \end{matrix}\right\}.$$

Thus the solutions to $z^n = 1$ are precisely $z = e^{2\pi i j/n}$ for $j \in \mathbb{Z}$. Since we know there can be no more than n solutions, there are necessarily redundancies in this list. Indeed, we have

$$e^{2\pi i j/n} = e^{2\pi i k/n} \;\Leftrightarrow\; \frac{2\pi j}{n} \equiv \frac{2\pi k}{n} \pmod{2\pi} \;\Leftrightarrow\; j \equiv k \pmod{n}.$$

It follows that there are exactly n *distinct* solutions, one for each congruence class modulo n. Choosing the standard remainder representatives for j, we arrive at the complete list of (distinct) solutions to $z^n = 1$ in $\mathbb{C}$:

$$\mu_n = \{e^{2\pi i j/n} \,|\, j = 0, 1, 2, \ldots, n-1\}.$$

Notice that if we let $\omega = e^{2\pi i/n}$, then $e^{2\pi i j/n} = \omega^j$. We therefore can also write

$$\mu_n = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}.$$

Let's summarize.

Theorem 1. *For $n \in \mathbb{N}$, $|\mu_n| = n$. In particular,*

$$\begin{aligned} \mu_n &= \{e^{2\pi i j/n} \,|\, j = 0, 1, 2, \ldots, n-1\} && (1) \\ &= \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}, && (2) \end{aligned}$$

where $\omega = e^{2\pi i/n}$.

[1] $z_1 \equiv z_2 \pmod{a}$ if and only if z_1 and z_2 map (under the canonical surjection) to the same coset in the quotient group $\mathbb{C}/a\mathbb{Z}$.

Remarks.

1. Equation (1) shows that, geometrically speaking, the elements of μ_n form the vertices of a regular n-gon on the unit circle.

2. The reader will readily verify that μ_n is a subgroup of $\mathbb{C}^\times$. The description (2) shows that μ_n is actually a *cyclic* group[2] of order n, and so is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

2 General nth Roots

The *nth roots of* $a \in \mathbb{C}^\times$ are the complex solutions of the equation $z^n = a$. Writing $z = re^{i\theta}$ and $a = Re^{i\phi}$ $(r, R > 0)$, we immediately find that $z^n = a$ if and only if $r^n = R$ and $n\theta \equiv \phi \pmod{2\pi}$. Rewriting these conditions as $r = \sqrt[n]{R}$ and $\theta \equiv \frac{\phi}{n} \pmod{\frac{2\pi}{n}}$, we find that a particular nth root of a is $\alpha = \sqrt[n]{R}e^{i\phi/n}$. The following theorem tells us that every other nth root differs from α by a factor in μ_n.

Theorem 2. *Let $n \in \mathbb{N}$ and $a \in \mathbb{C}^\times$. Then a has exactly n nth roots in $\mathbb{C}$, which are the members of the set*

$$\alpha\mu_n = \{\alpha, \alpha\omega, \alpha\omega^2, \ldots, \alpha\omega^{n-1}\}, \tag{3}$$

with $\alpha = \sqrt[n]{|a|}e^{i\frac{\operatorname{Arg} a}{n}}$.

Proof. Let A denote the set of nth roots of A. Suppose that $z = \alpha\zeta$ with $\zeta \in \mu_n$. Then

$$z^n = \alpha^n\zeta^n = a \cdot 1 = a,$$

so that $z \in A$. Thus $\alpha\mu_n \subset A$. Conversely, if $z \in A$ is an nth root of a, let $\zeta = z/\alpha$. Then $z = \alpha\zeta$ and

$$\zeta^n = \frac{z^n}{\alpha^n} = \frac{a}{a} = 1,$$

so that $\zeta \in \mu_n$. Hence $z \in \alpha\mu_n$ and $A \subset \alpha\mu_n$. This is enough to establish our result. □

Remarks.

3. Because the effect of multiplication by a complex number z is scaling by $|z|$ and (positive) rotation by $\arg z$, equation (3) shows that the nth roots of a also form the vertices of a regular n-gon, with radius $\sqrt[n]{|a|}$.

4. The conclusion of Theorem 2 still holds if α is replaced by *any* nth root of a, by the same proof.

5. The nth power map $P_n : \mathbb{C}^\times \to \mathbb{C}^\times$, given by $P_n(z) = z^n$, is a homomorphism with $\ker P_n = \mu_n$. The paragraph preceding the statement Theorem 2 proves that P_n is surjective, and Theorem 1 shows that $\ker P_n = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$. Theorem 2 is then just a restatement of the fact that the fiber over a point under a homomorphism is just a coset of the kernel.

3 Root Functions

The preceding section shows that in order to define $\sqrt[n]{z}$ as a *function* on $\mathbb{C}^\times$, a particular choice of nth roots needs to be made for each $z \in \mathbb{C}$. A similar situation occurs when working with real square roots, where one declares that for $x \in \mathbb{R}^+$, $\sqrt{x}$ denotes the *positive* square root of x. But rather than (directly) restricting the codomain in this manner in order to make $\sqrt[n]{z}$ a function, we will restrict the domain.

[2] This also follows from a much deeper result on finite multiplicative subgroups of fields.

In the course of proving Theorem 2 we were led to the relationship $\theta \equiv \frac{\phi}{n} \pmod{\frac{2\pi}{n}}$ between the arguments ϕ and θ of z and its nth roots, respectively. Because $2\pi\mathbb{Z}$ is a proper subgroup of $\frac{2\pi}{n}\mathbb{Z}$ (for $n \geq 2$), this congruence does not uniquely determine a single argument. This ambiguity is the source of the multi-valued nature of the nth root. However, as we've seen, if we simply replace the congruence by an equality we produce a single nth root. This, however, isn't sufficient to uniquely determine a *specific* nth root unless we also specify the value of ϕ, i.e. the argument of z.

Therefore, for $z \in \mathbb{C}^\times$ we define the *principal branch* of $\sqrt[n]{z}$ by

$$\sqrt[n]{z} = \sqrt[n]{|z|}e^{i\frac{\operatorname{Arg} z}{n}}.$$

If $\omega = e^{2\pi i/n}$, the functions

$$f_j(z) = \omega^j \sqrt[n]{z},\ j = 1, 2, \ldots, n-1$$

also define branches of the nth root function. Together with $\sqrt[n]{z}$, the values of these functions at a fixed z yield the n distinct nth roots of z.

Example. Let $z = re^{i\theta} \in \mathbb{C}^\times$. If $\theta \in (-\pi, \pi]$, then $\theta/2 \in (-\pi/2, \pi/2]$. In particular, $\cos\theta/2 \geq 0$. It follows from the half-angle formula for cosine that

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}.$$

Because θ and $\theta/2$ have the same sign, $\sin\theta$ and $\sin\theta/2$ also have the same sign. Therefore the half-angle formula for sine yields

$$\sin\frac{\theta}{2} = \operatorname{sgn}(\sin\theta)\sqrt{\frac{1-\cos\theta}{2}} =$$

Thus

$$\begin{aligned}\sqrt{z} = \sqrt{r}e^{i\theta/2} &= \sqrt{r}\left(\sqrt{\frac{1+\cos\theta}{2}} + i\operatorname{sgn}(\sin\theta)\sqrt{\frac{1-\cos\theta}{2}}\right)\\ &= \sqrt{\frac{r+r\cos\theta}{2}} + i\operatorname{sgn}(\sin\theta)\sqrt{\frac{r-r\cos\theta}{2}}\\ &= \sqrt{\frac{|z|+\operatorname{Re} z}{2}} + i\operatorname{sgn}(\operatorname{Im} z)\sqrt{\frac{|z|-\operatorname{Re} z}{2}}\end{aligned}$$

gives the real and imaginary parts of the principal branch of the square root.

4 Continuity of the Principal Branch of $\sqrt[n]{z}$

The distance function $d(z, w) = |z - w|$ turns $\mathbb{C}$ into a complete metric space. Because $d(z, w)$ is just the standard Euclidean distance when we identify $\mathbb{C}$ with $\mathbb{R}^2$, it induces the usual topology on $\mathbb{R}^2$. Because $\sqrt[n]{z} = \sqrt[n]{|z|}e^{i\frac{\operatorname{Arg} z}{n}}$, we can study its continuity by considering $|z|$ and $\operatorname{Arg} z$ separately.

The reverse triangle inequality implies that $|z|$ is continuous throughout $\mathbb{C}$. Hence so is $\sqrt[n]{|z|}$. Now consider $\operatorname{Arg} : \mathbb{C}^\times \to (-\pi, \pi]$. Let $I \subset (-\pi, \pi]$ be a relatively open interval. If $I = (\theta_1, \theta_2)$, then $\operatorname{Arg}^{-1}(I)$ is the open infinite sector $\theta_1 < \operatorname{Arg} z < \theta_2$. However, if $I = (\theta, \pi]$, then $\operatorname{Arg}^{-1}(I)$ is the half-open infinite sector $\theta < \operatorname{Arg} z \leq \pi$. If $\theta \in (-\pi, \pi)$, this is sector is not open in $\mathbb{C}^\times$. Thus $\operatorname{Arg} z$ fails to be a continuous function.

$\operatorname{Arg} z$ fails to be continuous simply because the endpoint π belongs to its codomain. We can omit the value π provided that we also delete $\operatorname{Arg}^{-1}(\{\pi\}) = (-\infty, 0)$ from $\mathbb{C}^\times$. This produces the *slit plane* $\Omega = \mathbb{C}\backslash(-\infty, 0]$, and we immediately conclude that

$$\operatorname{Arg} : \Omega \to (-\pi, \pi)$$

is continuous. Because $\sqrt[n]{|z|}$ is continuous everywhere, we conclude that $\sqrt[n]{z}$ becomes a continuous function when restricted to the slit plane Ω. We cannot include the slit because for $x \in (-\infty, 0)$ we have

$$\sqrt[n]{x + i0^+} = \lim_{y \to 0^+} \sqrt[n]{x + iy} = \sqrt[n]{-x} e^{i\pi/n},$$
$$\sqrt[n]{x + i0^-} = \lim_{y \to 0^-} \sqrt[n]{x + iy} = \sqrt[n]{-x} e^{-i\pi/n},$$

since as we approach the negative real axis from above, $\operatorname{Arg} z \to \pi$, whereas $\operatorname{Arg} z \to -\pi$ as we approach from below. Consequently $\lim_{z \to x} \sqrt[n]{z}$ does not exist for $x \in (-\infty, 0)$.