Second Order Linear ODEs w/ Constant Coeffs.

$$ay'' + by' + cy = G(x) \quad (*)$$

(a, b, c constants)

If $G(x) \equiv 0$, we call $(*)$ **homogeneous**.

Ex: 1. $y'' - y' - 6y = 0$

2. $y'' - 4y' + 13y = 0$

homog.

3. $y'' - 2y' - 2y = e^x$

4. $y'' - 4y' + 4y = 3\sin x$

inhomog.

Theorem: (Principle of Superposition)

If y_1, y_2 are solutions to

$$ay'' + by' + cy = 0,$$

then so is $y = c_1 y_1 + c_2 y_2$ where c_1, c_2 are any constants.

If y_1 and y_2 are not multiples of one another, then the general sol'n is given by

$$y = c_1 y_1 + c_2 y_2.$$

Remark: When y_1, y_2 are not multiples of each other, we say

$$\{y_1, y_2\}$$

is linearly independent or a fund. set.

Ex: $y'' - y' - 6y = 0$

Claim: $y_1 = e^{-2x}$, $y_2 = e^{3x}$

are both sol'ns.

Why: $y_1'' - y_1' - 6y_1 = 4e^{-2x} - (-2e^{-2x})$

$$- 6e^{-2x}$$

$$= 0 \checkmark$$

$$y_2'' - y_2' - 6y_2 = 9e^{3x} - 3e^{3x}$$

$$- 6e^{3x}$$

$$= 0 \checkmark$$

Claim: y_1, y_2 are lin. ind.

Why: Suppose not:

$$\cancel{y_1 = C y_2}$$

$$e^{-2x} = Ce^{3x}$$

$\cancel{C = e^{-5x}}$

Superposition $\Rightarrow$ Gen. sol'n is

$$\boxed{y = c_1 e^{-2x} + c_2 e^{3x}}$$

$\square$

$y_1 = e^{-2x}$, $y_2 = 7e^{-2x}$

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{-2x} + 7c_2 e^{-2x} \\ &= (c_1 + 7c_2) e^{-2x} \\ &= Ce^{-2x} \end{aligned}$$

? $y = e^{3x}$

Question 1: How did we find e^{-2x}, e^{3x}?

Question 2: When will

$$ay'' + by' + cy = 0$$

have $y = e^{rx}$ as a solution?

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$(ar^2 + br + c)\cancel{e^{rx}} = 0$$

$$\boxed{ar^2 + br + c = 0}$$

↳ <u>Characteristic Eqn.</u> of $ay'' + by' + cy = 0$

Theorem: If $ar^2 + br + c = 0$ has two real roots $r_1 \neq r_2$ $(b^2 - 4ac > 0)$, then the general solution to

$$ay'' + by' + cy = 0$$

is given by

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Ex: 1. $y'' - 2y' - 2y = 0$

$\Big\downarrow$ char. eqn.

$r^2 - 2r - 2 = 0$

$$\Rightarrow \quad r = \frac{2 \pm \sqrt{4 - 4(-2)}}{2}$$

$$= \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$$

$\Rightarrow$ Gen. sol'n is

$$\boxed{y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}}$$

∎

2. $2y'' + 3y' + y = 0$

$\Big\downarrow$ char. eqn.

$2r^2 + 3r + 1 = 0$

$(2r+1)(r+1) = 0$

$$r = -1, -1/2$$

Gen. Sol'n:

$$\boxed{y = c_1 e^{-x} + c_2 e^{-x/2}}$$

3. $y'' - 4y' + 4y = 0$

$\}$ char. eqn.

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

Problem: $r = 2 \rightsquigarrow y_1 = e^{2x}$

Need a second independent sol'n...

Idea: Try $y_2 = xe^{2x}$:

$$y_2' = e^{2x} + 2xe^{2x} = (1+2x)e^{2x}$$

$$y_2'' = 2e^{2x} + 2(1+2x)e^{2x}$$

$$= (4+4x)e^{2x}$$

$$y_2'' - 4y_2' + 4y_2 = (\cancel{4} + \cancel{4x})e^{2x}$$

$$-4(\cancel{1} + \cancel{2x})e^{2x} + \cancel{4}xe^{2x}$$

$$= 0 \ !$$

So $y_2 = xe^{2x}$ is another sol'n!

$y_1 = e^{2x}$, $y_2 = xe^{2x}$ are ind., so:

Gen sol'n is:

$$\boxed{\begin{aligned} y &= c_1 e^{2x} + c_2 x e^{2x} \\ &= (c_1 + c_2 x)e^{2x} \end{aligned}}$$

∎