Homogeneous Second Order Linear ODEs

$$ay'' + by' + cy = 0$$

$\Big\}$ char. eqn.

$$ar^2 + br + c = 0 \qquad (*)$$

Case 1: $(*)$ has two real roots $r_1 \neq r_2$

$\Rightarrow$ Gen. sol'n to ODE is

$b^2 - 4ac > 0$

$$\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

Case 2: $(*)$ has only <u>one</u> (repeated) root r

$\Rightarrow$ Gen. sol'n to ODE is

$b^2 - 4ac = 0$

$$y = (c_1 + c_2 x) e^{rx}$$
$$= c_1 e^{rx} + c_2 x e^{rx}$$

Ex: Solve the IVP

$$9y'' - 6y' + y = 0$$

$$y(0) = 1$$

$$y'(0) = -1$$

Sol'n: Char. eqn.:

$$9r^2 - 6r + 1 = 0$$

$$(3r-1)^2 = 0$$

$$r = 1/3$$

So gen. sol'n is

$$y = (c_1 x + c_2)e^{x/3}$$

Init. conds.:

$$1 = y(0) = c_2 e^0 = c_2 \Rightarrow c_2 = 1$$

$$-1 = y'(0) = c_1 e^{x/3} + \frac{1}{3}(c_1 x + c_2)e^{x/3}\Big|_{x=0}$$

$$= c_1 + \frac{c_2}{3} = c_1 + \frac{1}{3}$$

$$\Rightarrow \quad c_1 = -1 - \frac{1}{3} = -\frac{4}{3}$$

Sol'n to IVP:

$$\boxed{y = \left(-\frac{4}{3}x + 1\right)e^{x/3}}$$

$\blacksquare$

<u>Case 3</u>: (*) has <u>nonreal</u> complex sol'ns...

$\hookrightarrow \quad b^2 - 4ac < 0$

$$r = \alpha \pm i\beta, \quad \beta \neq 0$$

<u>Expect</u> to get "sol'ns":

$$y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}$$

$$y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}$$

<u>Question</u>: What does $e^{\text{imag} \#}$ <u>mean</u>?

Ex: Consider $y'' + y = 0 \Leftrightarrow y'' = -y$

char. eqn.

$r^2 + 1 = 0 \Rightarrow r = \pm\sqrt{-1} = \pm i$

1st Attempt: $y_1 = e^{ix}$, $y_2 = e^{-ix}$

are the fund. sol'ns to ODE

?!

2nd Attempt: Notice that

$y_1 = \cos x$, $y_2 = \sin x$

both solve the ODE!

Princ. of Superpos. $\Rightarrow$ Gen sol'n. is

$y_1 = C_1 \cos x + C_2 \sin x$.

We will use this to "explain" e^{ix}.

Since $y = e^{ix}$ is a "sol'n" to

the ODE, we expect

$$\boxed{y = e^{ix} = c_1 \cos x + c_2 \sin x}$$

for <u>some</u> c_1, c_2.

$\underline{x = 0}$: $\quad e^0 = c_1 \cos 0 + c_2 \sin 0$

$$1 = c_1$$

$$y' = i e^{ix} = -c_1 \sin x + c_2 \cos x$$

$$i e^0 = -c_1 \cdot \sin 0 + c_2 \cdot \cos 0$$

$$i = c_2$$

<u>Conclusion</u>: (Euler's Formula)

$$\boxed{e^{ix} = \cos x + i \sin x}$$

<u>Remarks</u>:

1. $e^{-ix} = e^{i(-x)}$

$$= \underline{\cos(-x)} + i\,\underline{\sin(-x)}$$

$$= \cos(x) - i\sin(x)$$

2. A little algebra shows:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Back to Earlier Question:

$$ay'' + by' + cy = 0$$

$$\Big\downarrow \text{ char. eqn.}$$

$$ar^2 + br + c = 0$$

$b^2 - 4ac < 0 \Rightarrow$ Roots are $r = \alpha \pm i\beta$
$(\beta \neq 0)$

This yields the "sol'ns"

$$\tilde{y}_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}$$

$$\tilde{y}_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}$$

By Princ. of Superpos., we <u>also</u> have sol'ns:

$$y_1 = \frac{1}{2}\tilde{y}_1 + \frac{1}{2}\tilde{y}_2$$

$$= e^{\alpha x}\left(\frac{e^{i\beta x} + e^{-i\beta x}}{2}\right)$$

$$= e^{\alpha x}\cos\beta x \leftarrow \text{Real-valued!}$$

$$y_2 = \frac{1}{2i}\tilde{y}_1 - \frac{1}{2i}\tilde{y}_2$$

$$= e^{\alpha x}\left(\frac{e^{i\beta x} - e^{-i\beta x}}{2i}\right)$$

$$= e^{\alpha x}\sin\beta x \leftarrow \text{Real-valued!}$$

$\Rightarrow$ Gen. sol'n to ODE is

$$\boxed{y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)}$$