Homogeneous Second Order Linear ODEs (Concl.)

$$ay'' + by' + cy = 0$$

char. eqn.

$$ar^2 + br + c = 0$$

exp. sol'ns

$$y = e^{rx}$$

When $ar^2 + br + c = 0$ has <u>nonreal</u> complex roots, we were led to ask:

$$e^{\text{imag. \#}} = ?$$

<u>Euler's Formula</u>: $e^{ix} = \cos x + i \sin x$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

Case 3: If $ar^2+br+c=0$ has complex roots $r = \alpha \pm i\beta$ $(\beta \neq 0)$, then $ay''+by'+cy=0$ has the fund. sol'ns

$(b^2-4ac<0)$

$$y_1 = e^{\alpha x}\cos\beta x, \quad y_2 = e^{\alpha x}\sin\beta x,$$

so that the gen. sol'n is

$$y = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)$$

Ex: 1. Solve $16y''-8y'+145y=0$.

char. eqn.

$$16r^2-8r+145=0$$

Quad. Formula: $r = \dfrac{8 \pm \sqrt{64-4\cdot16\cdot145}}{2\cdot16}$

$$= \frac{\cancel{8} \pm \cancel{8}\sqrt{-144}}{2\cdot\cancel{16}\,2} = \frac{1 \pm 12i}{4}$$

$$= \underbrace{\frac{1}{4}}_{\alpha} \pm \underbrace{3i}_{\beta}$$

Gen. sol'n is:

$$\boxed{y = e^{x/4}\left(c_1 \cos 3x + c_2 \sin 3x\right).}$$

$\blacksquare$

2. Solve $y'' - 4y' + 13y = 0$

$$\Big\downarrow \text{ char. poly.}$$

$$r^2 - 4r + 13 = 0$$

Quad. Formula:

$$r = \frac{4 \pm \sqrt{4(4-13)}}{2}$$

$$= \frac{4 \pm 2\sqrt{4-13}}{2}$$

$$= 2 \pm 3i$$

$$\underbrace{2}_{\alpha} \pm \underbrace{3}_{\beta} i$$

Gen. sol'n:

$$\boxed{y = e^{2x}\left(c_1 \cos 3x + c_2 \sin(3x)\right)}$$

∎

3. Solve the IVP

$$y'' - 2y' + 5y = 0$$

$$y(\pi) = 0, \quad y'(\pi) = 2$$

Char. Eqn.: $r^2 - 2r + 5 = 0$

$$\Rightarrow \quad r = \frac{2 \pm \sqrt{4 - 4 \cdot 5}}{2}$$

$$= 1 \pm \sqrt{1-5} = 1 \pm 2i$$

Gen. Sol'n: $y = e^{x}\left(\cancel{c_1 \cos 2x} + c_2 \sin 2x\right)$

Init. Conds:

$$0 = y(\pi) = e^{\pi}\left(c_1 \overset{1}{\cancel{\cos 2\pi}} + c_2 \overset{0}{\cancel{\sin 2\pi}}\right)$$

$$0 = c_1 e^{\pi}$$

$$\Rightarrow c_1 = 0$$

$$2 = y'(\pi) = c_2\left(e^x \sin 2x + 2e^x \cos 2x\right)\Big|_{x=\pi}$$

$$= c_2\left(e^{\pi} \overset{0}{\cancel{\sin 2\pi}} + 2e^{\pi} \overset{1}{\cancel{\cos 2\pi}}\right)$$

$$\cancel{2} = \cancel{2} c_2 e^{\pi} \Rightarrow c_2 = e^{-\pi}$$

Sol'n:

$$\boxed{\begin{aligned} y &= e^x\left(e^{-\pi} \sin 2x\right) \\ &= e^{x-\pi} \sin 2x \end{aligned}}$$

4. Solve the boundary value problem

$y(x_0) = A$ $\qquad$ $y'' + 100y = 0$

$y'(x_0) = B$ $\qquad$ $y(\underline{\underline{0}}) = 2, \quad y(\underline{\underline{\pi}}) = 5,$

if the solution exists.

Sol'n: Char. eqn.: $r^2 + 100 = 0$

$r^2 = -100$

$r = \pm 10i$

$= \underbrace{0}_{\alpha} \pm \underbrace{10}_{\beta} i$

Gen. Sol'n:

$$y = c_1 \cos 10x + c_2 \sin 10x$$

Boundary Conds.:

$$2 = y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1$$

$$5 = y(\pi) = c_1 \cos 10\pi + c_2 \sin 10\pi = c_1$$

Since $2 \neq 5$, there is

$$\boxed{\text{no solution}}$$

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Inhomogeneous Second Order Linear ODEs

(1) $\quad ay'' + by' + cy = G(x)$

nonzero

The complementary eqn. to (1) is

(2) $\quad ay'' + by' + cy = 0.$

Theorem: If y_p is a particular sol'n to (1) and $c_1 y_1 + c_2 y_2$ is the gen. sol'n to (2), then the gen. sol'n to (1) is

$$y = y_p + c_1 y_1 + c_2 y_2.$$

Ex: Solve $y'' + y' - 2y = x^2 + 3$.

1. Solve comp. eqn. $y'' + y' - 2y = 0$:

char. eqn.

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0 \Rightarrow r = 1, -2$$

$$\boxed{y = c_1 e^x + c_2 e^{-2x}}$$

2. Find <u>some</u> sol'n to $y'' + y' - 2y = x^2 + 3.$

We "guess"

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' + y_p' - 2y_p = \boxed{-2Ax^2 + (2A-2B)x + (2A+B-2C)}$$

$$= \boxed{x^2 + 3}$$

$$\Rightarrow \quad -2A = 1 \quad \Rightarrow A = -1/2$$

$$2A - 2B = 0 \quad \Rightarrow B = -1/2$$

$$2A + B - 2C = 3 \quad \Rightarrow C = -9/4$$

$$\Rightarrow \boxed{y_p = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{9}{4}}$$

is a sol'n to the given ODE.

3. Combine: The general sol'n to the (inhomog.) eqn. is

$$\boxed{y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{9}{4} + c_1 e^x + c_2 e^{-2x}}$$

$\blacksquare$