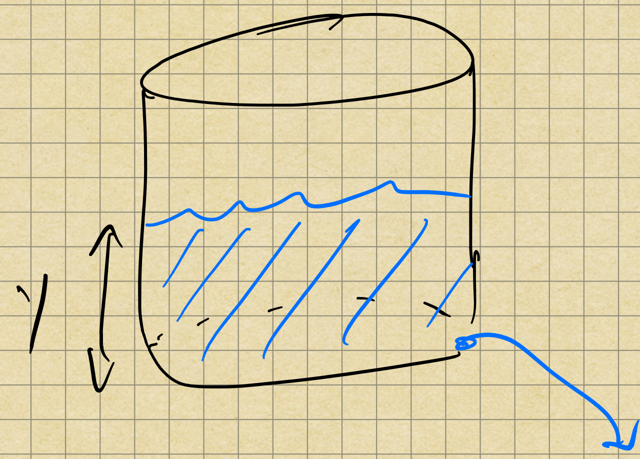
## F16, #5:




$$\frac{dy}{dt} = -c\sqrt{y}$$

$y(0) = 4$

$y(2) = 1$

$y(t) = 0$ ?

**Solve ODE:**

$$\int y^{-1/2}\,dy = \int -c\,dt$$

$$\frac{y^{1/2}}{1/2} = -ct + k$$

$$y^{1/2} = \frac{k - ct}{2}$$

$$4^{1/2} = y(0)^{1/2} = \frac{k}{2} \Rightarrow k = 4$$

$$1^{1/2} = y(2)^{1/2} = \frac{4 - 2c}{2} = 2 - c$$

$$\Rightarrow c = 1$$

$$y^{1/2} = \frac{4 - t}{2}$$

$$y = \left(\frac{4 - t}{2}\right)^2 = 0 \; ?$$

$$\boxed{t = 4}$$

$\blacksquare$

$$\int_{-1}^{0} \frac{x^3-4x+1}{x^2-3x+2}\,dx$$

Long Division:

$$
\begin{array}{r|l}
 & \;x+3 \\
\hline
x^2-3x+2 & \;\cancel{x^3}-4x+1 \\
 & -\cancel{x^3}+3x^2-2x \\
\hline
 & \;3\cancel{x^2}-6x+1 \\
 & -\cancel{3x^2}+9x-6 \\
\hline
 & \;3x-5
\end{array}
$$

$$\frac{x^3-4x+1}{x^2-3x+2} = x+3+\frac{3x-5}{x^2-3x+2}$$

Partial Fractions:

$$\frac{3x-5}{x^2-3x+2} = \frac{3x-5}{(x-2)(x-1)}$$

$$= \frac{A}{x-2} + \frac{B}{x-1}$$

$$A = \frac{3\cdot 2-5}{\cancel{(2-2)}(2-1)} = 1$$

$$B = \frac{3\cdot 1-5}{(1-2)\cancel{(1-1)}} = \frac{-2}{-1} = 2$$

Integrate:

$$\int \frac{x^3-4x+1}{x^2-3x+2}\,dx$$

$$= \int x+3 + \frac{1}{x-2} + \frac{2}{x-1}\,dx$$

$$= \boxed{\frac{x^2}{2} + 3x + \ln|x-2| + 2\ln|x-1| + C}$$

$$\frac{dy}{dx} = x + xy^2 - 3y^2 - 3, \quad y(0) = 1$$

Separable

~~First Order Linear~~

~~Second Order Linear~~

$$\frac{dy}{dx} = x(1+y^2) - 3(1+y^2)$$

$$= (x-3)(1+y^2)$$

$$\int \frac{dy}{1+y^2} = \int x-3 \; dx$$

$$\arctan y = \frac{x^2}{2} - 3x + C$$

$\underline{y(0)=1}$: $$\arctan 1 = \frac{0}{2} - 3\cdot 0 + C$$

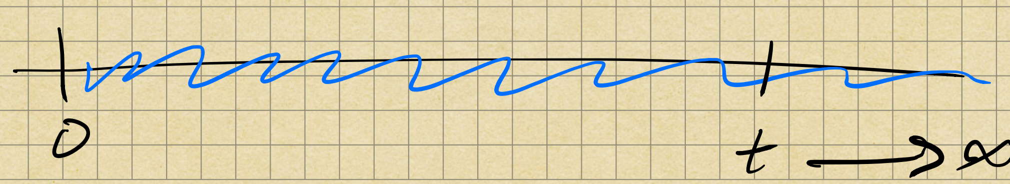
$$\frac{\pi}{4} = C$$

$$\arctan y = \frac{x^2}{2} - 3x + \frac{\pi}{4}$$

$$\boxed{y = \tan\left(\frac{x^2}{2} - 3x + \frac{\pi}{4}\right)}$$

∎

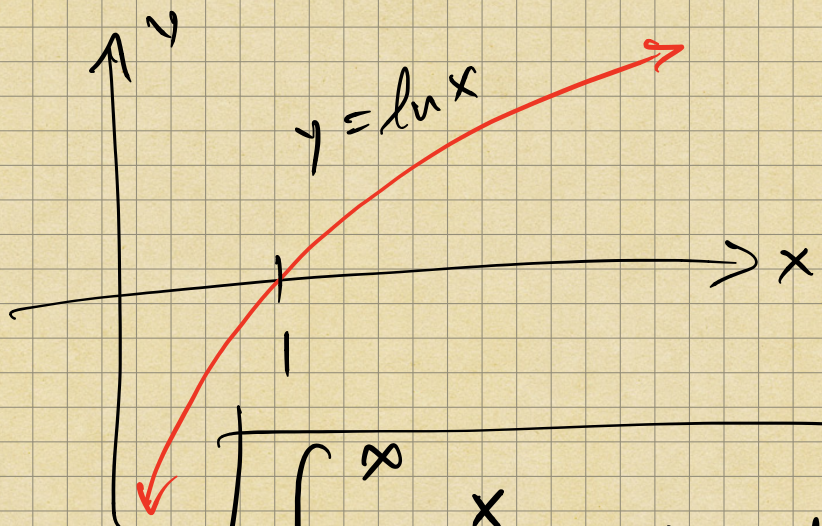
$$\int_0^\infty f(x)\,dx = \lim_{t\to\infty} \int_0^t f(x)\,dx$$



$$\int_0^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_0^t \frac{x}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln(x^2+1)\Big|_0^t$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln(t^2+1) = \infty$$




$$\boxed{\int_0^\infty \frac{x}{x^2+1}\,dx \text{ diverges}}$$

or ?

$$\boxed{\int_0^\infty \frac{x}{x^2+1}\,dx = \infty}$$

Limit Rules for $0, \infty$:

$\infty + \infty = \infty$

$\infty - \infty$ is undefined

$\frac{\infty}{\infty}$ is undefined

$0 \cdot \infty$ is undefined

$\frac{0}{0}$ is undefined

$\frac{1}{0}$ is undefined

$$\boxed{\frac{1}{\infty} = 0, \quad \frac{1}{0^+} = \infty, \quad \frac{1}{0^-} = -\infty}$$

$$xy' - 2y = x^2, \quad x > 0$$

mult. by $\frac{1}{x}$ $\Big\downarrow$ "standard" form

$$y' - \frac{2}{x}y = x$$

$$\mu = e^{\int -\frac{2}{x}\,dx}$$

$$= e^{-2\ln x} = (e^{\ln x})^{-2}$$

$$= x^{-2}$$

$$x^{-2}y' - 2x^{-3}y = x^{-1}$$

$$(x^{-2}y)' = x^{-1}$$

$$x^{-2}y = \int x^{-1}\,dx$$

$$= \ln x + C$$

$$\boxed{y = x^2(\ln x + C)}$$

$\blacksquare$

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$$\int \frac{dx}{x^2+1} = \boxed{\arctan x + C}$$

$$\downarrow$$

$$\int \frac{dx}{(x+i)(x-i)} = \int \frac{\frac{1}{2i}}{x-i} - \frac{\frac{1}{2i}}{x+i}\,dx$$

$$= \frac{1}{2i}\left(\ln(x-i) - \ln(x+i)\right) + C$$

$$= \boxed{\frac{1}{2i}\ln\left(\frac{x-i}{x+i}\right) + C}$$

F13, #4:

$$25y'' + by' + cy = 0$$

Gen. Sol'n:

$$\boxed{y = e^{x/5}\left(c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}\,x\right)}$$

Find b, c...

Roots of char. eqn. are

$$r = \frac{1}{5} \pm i\sqrt{2}$$

Char. eqn. itself is

$$25r^2 + br + c = 0$$

Option 1: Plug $r = \frac{1}{5} + i\sqrt{2}$ into and solve for $b, c$.

Option 2: Factor the char. poly. as:

$$25\left(r - \left(\tfrac{1}{5} + i\sqrt{2}\right)\right)\left(r - \left(\tfrac{1}{5} - i\sqrt{2}\right)\right)$$

$$= 25\left(r^2 - \frac{2}{5}r + \frac{1}{25} + 2\right)$$

$$= 25r^2 - 10r + 51$$

$$\Rightarrow b = -10, \quad c = 51$$

$$ay'' + by' + cy = 0$$

$\Big\updownarrow$ char. equ.

$$ar^2 + br + c = 0$$

Case 1 : Roots $r_1 \neq r_2$ (real) :

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

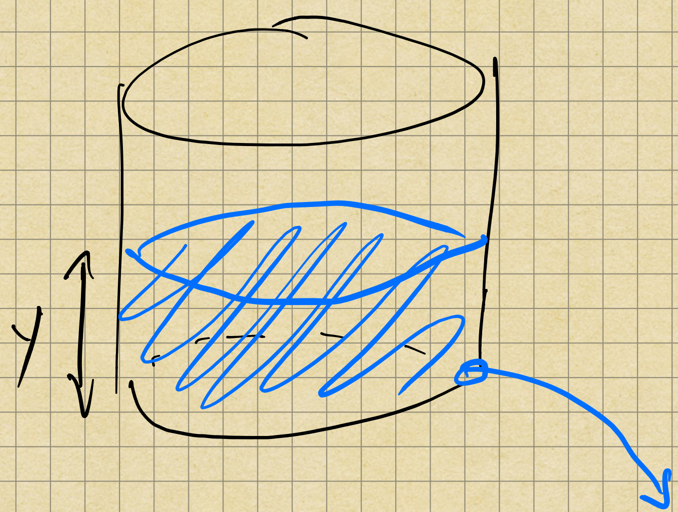
Case 2 : Root $r$ (repeated)

$$y = (c_1 + c_2 x) e^{rx}$$

Case 3 : Roots $r = \alpha \pm \beta i$ $(\beta \neq 0)$

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

F13, #5:




$$\frac{dy}{dt} = -c\sqrt{y}$$

$y(0) = 4$

$y(2) = 1$

$y(t) = 0$ ?

Solve ODE:

$$\int \frac{dy}{\sqrt{y}} = \int -c\,dt$$

$$\int y^{-1/2}\,dy = \int -c\,dt$$

$$\frac{y^{1/2}}{1/2} = -ct + k$$

$$2y^{1/2} = -ct + k$$

$$y^{1/2} = \frac{-ct+k}{2}$$

$\underline{y(0)=4}$: $\quad 4^{1/2} = \dfrac{-c \cdot 0 + k}{2}$

$$2 = \frac{k}{2} \Rightarrow k = 4$$

$\underline{y(2)=1}$ : $1^{1/2} = \dfrac{-c\cdot 2+4}{2}$

$$1 = -c+2$$

$$c = 1$$

$$\Rightarrow \quad y^{1/2} = \frac{4-t}{2}$$

$\underline{y(t)=0}$ : $0^{1/2} = \dfrac{4-t}{2}$

$$0 = \frac{4-t}{2}$$

$$\boxed{t=4}$$

$\blacksquare$

F13, #6 :

$$\frac{dP}{dt} = P - ht$$

a) Solve for $P$ in terms of $P(0) = P_0$ and $h$.

$$\left(\frac{dP}{dt} - P = -ht\right) \leftarrow \text{Linear, } 1^{st} \text{ order}$$

$$\mu = e^{\int -dt} = e^{-t}$$

$$e^{-t}\frac{dP}{dt} - e^{-t}P = -hte^{-t}$$

$$\left(e^{-t}P\right)' = -hte^{-t}$$

$$e^{-t}P = \int -hte^{-t}\,dt$$

$$= -h\int te^{-t}\,dt$$

| | |
|---|---|
| $t$ | $e^{-t}$ |
| $1$ | $-e^{-t}$ |
| $0$ | $e^{-t}$ |

$$e^{-t}P = -h\left(-te^{-t} - e^{-t}\right) + C$$

$$= hte^{-t} + he^{-t} + C$$

$P(0) = P_0$

$$e^{-0}P_0 = h \cdot 0 + he^{0} + C$$

$$P_0 = h + C$$

$$\Rightarrow \quad C = P_0 - h$$

$$e^{-t}P = hte^{-t} + he^{-t} + (P_0 - h)$$

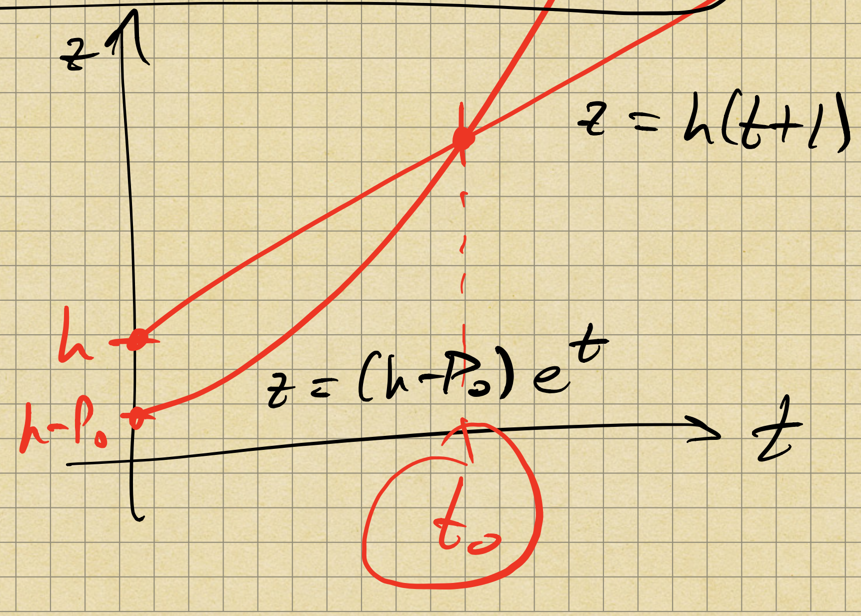
$$P = ht + h + (P_0 - h)e^{t}$$

(b) Show $P = 0$ eventually if $h > P_0$.

$$0 = P(t)$$

$$0 = h(t+1) + (P_0 - h)e^{t}$$

$$\boxed{(h - P_0)e^{t} = h(t+1)}$$



# Arithmetic of $\infty$:

$$\left.\begin{array}{c} \infty - \infty \\ \\ \dfrac{\infty}{\infty} \\ \\ \dfrac{0}{0} \\ \\ \infty \cdot 0 \\ \\ \dfrac{1}{0} \end{array}\right\} \text{Undefined}$$

$$\frac{1}{\infty} = 0, \quad \frac{1}{0^+} = \infty, \quad \frac{1}{0^-} = -\infty$$

$$\infty + \infty = \infty$$

$$\infty \cdot \infty = \infty$$

$$\int_{-1}^{0} \frac{x^3 - 4x + 1}{x^2 - 3x + 2}\, dx$$

Long Division :

$$\begin{array}{r|l}
 & x + 3 \\
\hline
x^2 - 3x + 2 & x^3 - 4x + 1 \\
 & -(x^3 - 3x^2 + 2x) \\
\hline
 & 3x^2 - 6x + 1 \\
 & -(3x^2 - 9x + 6) \\
\hline
 & 3x - 5
\end{array}$$

$$\frac{x^3 - 4x + 1}{x^2 - 3x + 2} = x + 3 + \frac{3x - 5}{x^2 - 3x + 2}$$

Partial Fractions :

$$\frac{3x-5}{x^2-3x+2} = \frac{3x-5}{(x-1)(x-2)}$$

$$= \frac{A}{x-1} + \frac{B}{x-2}$$

$x=1$

$x=2$

$$A = \frac{3\cdot 1-5}{\cancel{(1-1)}(1-2)} = \frac{-2}{-1} = 2$$

$$B = \frac{3\cdot 2-5}{(2-1)\cancel{(2-2)}} = \frac{1}{1} = 1$$

Integrate:

$$\int_{-1}^{0} x+3 + \frac{2}{x-1} + \frac{1}{x-2}\, dx$$

$$= \left.\frac{x^2}{2} + 3x + 2\ln|x-1| + \ln|x-2|\right|_{-1}^{0}$$

$$= \ln 2 - \left(\frac{1}{2} - 3 + 2\ln 2 + \ln 3\right)$$

$$= \frac{5}{2} - \ln 2 - \ln 3$$

$$= \boxed{\frac{5}{2} - \ln 6}$$