Method of Undetermined Coefficients

To solve

$$ay'' + by' + cy = G(x)$$

we:

1. Solve the complementary eqn.

$$ay'' + by' + cy = 0$$

to get $y_h = c_1 y_1 + c_2 y_2$.

2. Find one particular sol'n to

$$ay'' + by' + cy = G(x)$$

somehow, y_p.

3. The gen. sol'n is

$$y = y_p + y_h = y_p + c_1 y_1 + c_2 y_2.$$

Ex: Find one sol'n to

$$y'' - 4y' + 5y = (x+2)e^x$$

Sol'n: We guess

$$y = (Ax+B)e^x$$

$$y' = Ae^x + (Ax+B)e^x$$
$$= (Ax + A + B)e^x$$

$$y'' = Ae^x + (Ax+A+B)e^x$$
$$= (Ax + 2A + B)e^x$$

Plug into ODE:

Some alg.

$$y'' - 4y' + 5y = \boxed{\begin{aligned}&(2Ax + (2B-2A))e^x \\ &= (x+2)\,e^x\end{aligned}}$$

To get sol'n we need

$$2A = 1 \implies A = 1/2$$

$$2B - 2A = 2 \implies B = 3/2$$

So $\boxed{y_p = \left(\frac{1}{2}x + \frac{3}{2}\right)e^x}$ ■

Ex: Find <u>one</u> sol'n to

$$9y'' - 6y' + y = \cos 2x.$$

Sol'n: We guess:

$$y = A\cos 2x + B\sin 2x$$
$$y' = -2A\sin 2x + 2B\cos 2x$$
$$y'' = -4A\cos 2x - 4B\sin 2x$$

Plug into ODE:

Some alg.

$$9y'' - 6y' + y = (-35A - 12B)\cos 2x + (12A - 35B)\sin 2x \overset{\text{want}}{=} \cos 2x$$

We need:

$$\begin{aligned} -35A - 12B &= 1 \\ 12A - 35B &= 0 \end{aligned} \implies \begin{aligned} A &= -35/1369 \\ B &= -12/1369 \end{aligned}$$

So

$$\boxed{y_p = \frac{-35}{1369}\cos 2x - \frac{12}{1369}\sin 2x}$$

$\blacksquare$

<u>Ex</u>: Find <u>one</u> sol'n to

$$y'' + y' - 2y = e^x \sin x.$$

<u>Sol'n</u>: We guess:

$$y = e^x (A\cos x + B\sin x)$$

$$y' = e^x \big((A+B)\cos x + (B-A)\sin x \big)$$

$$y'' = e^x \big(2B\cos x - 2A\sin x \big)$$

Plug into ODE:

alg.

$$y'' + y' - 2y = e^x \big((3B - A)\cos x + (-3A - B)\sin x \big)$$

want

$$= e^x \sin x$$

So we need:

$$3B - A = 0 \qquad\qquad A = -3/10$$
$$\Rightarrow$$
$$-3A - B = 1 \qquad\qquad B = -1/10$$

So we have

$$\boxed{y_p = e^x\left(-\frac{3}{10}\cos x - \frac{1}{10}\sin x\right)}$$

☒

Q: Can this approach ever fail?

Ex: Find one sol'n to $y'' - y = e^{-x}$.

Soln: We guess:

$$\begin{cases} y = Ae^{-x} \\ y' = -Ae^{-x} \\ y'' = Ae^{-x} \end{cases}$$

Plug into ODE:

$$y'' - y = Ae^{-x} - Ae^{-x} = 0 \;!$$

$$= e^{-x}$$

Cannot agree!

want

Problem: The "guess" was actually a soln of the complementary eqn.!

Modification: Mult. original guess by x:

$$y = Axe^{-x}$$

$$y' = (-Ax + A)e^{-x}$$

$$y'' = (Ax - 2A)e^{-x}$$

Plug into ODE: alg.

$$y'' - y = -2Ae^{-x} = e^{-x} \implies -2A = 1$$

want

$$A = -1/2$$

So we get $\boxed{y_p = -\frac{1}{2}xe^{-x}}$ ☒

Moral: If "guess" involves terms that are solns of the compl. eqn., then we mult. our guess by x first.

Repeat if necessary.

<u>Ex</u>: Find the gen. sol'n to

$$y'' + 4y = x\cos 2x$$

<u>Sol'n</u>: 1. Solve compl. eqn.

$$y'' + 4y = 0$$

$$\Big\} \text{ char. eqn.}$$

$$r^2 + 4 = 0 \Rightarrow r^2 = -4$$

$$\Rightarrow r = \pm 2i$$

$$\Rightarrow y_h = c_1 \cos 2x + c_2 \sin 2x.$$

2. Find <u>one</u> sol'n to orig. ODE by "guessing":

$$y = (Ax + B)\cos 2x + (Cx + D)\sin 2x$$

Problematic terms

mult. by x

$$y = (Ax^2 + Bx)\cos 2x + (Cx^2 + Dx)\sin 2x$$

Plug into ODE to get

$$y'' + 4y = (\underline{8Cx + 2A + 4D})\cos 2x$$
$$+ (\underline{-8Ax - 4B + 2C})\sin 2x$$

alg.

$$= x\cos 2x$$

want

So we need:

$$-8A = 0 \qquad 8C = 1$$

$$-4B + 2C = 0 \qquad 2A + 4D = 0$$

(alg.) $\Rightarrow$ $A = D = 0$, $B = 1/16$, $C = 1/8$

So the sol'n is:

$$\boxed{y = \underbrace{\frac{1}{16}x\cos 2x + \frac{1}{8}x^2\sin 2x}_{y_p} + c_1\cos 2x + c_2\sin 2x}$$

$\blacksquare$

Method of Undetermined Coefficients

To solve the inhomog. eqn.

$$ay'' + by' + cy = G(x)$$

1. Solve the complementary eqn.

$$ay'' + by' + cy = 0$$

to get $y_h = c_1 y_1 + c_2 y_2$.

2. Find one particular sol'n to

$$ay'' + by' + cy = G(x)$$

by any method y_p.

3. To get gen. sol'n to inhomog eqn., just add:

$$y = y_p + y_h = y_p + c_1 y_1 + c_2 y_2$$

Ex: Find one sol'n to

$$y'' - 4y' + 5y = (x+2)e^x$$

Sol'n: We "guess"

$$y = (Ax+B)e^x$$
$$y' = (Ax + A + B)e^x$$
$$y'' = (Ax + 2A + B)e^x$$

Now plug into ODE: alg.

$$y'' - 4y' + 5y = (2Ax + 2B - 2A)e^x$$

want $\longrightarrow$ $= (x+2)e^x$ equal

So we need:

$$2A = 1 \implies A = 1/2$$
$$2B - 2A = 2 \implies B = 3/2$$

This yields

$$\boxed{y_p = \left(\frac{1}{2}x + \frac{3}{2}\right)e^x}$$

Ex: Find one sol'n to

$$9y'' - 6y' + y = \cos 2x.$$

Sol'n: We "guess"

$$\begin{cases} y = A\cos 2x + B\sin 2x \\ y' = -2A\sin 2x + 2B\cos 2x \\ y'' = -4A\cos 2x - 4B\sin 2x \end{cases}$$

Plug into ODE:

alg.

$$9y'' - 6y' + y = (-35A - 12B)\cos 2x + \underline{(12A - 35B)}\sin 2x$$

want $\longrightarrow = \cos 2x$

So we need:

$$\begin{aligned} 12A - 35B &= 0 \\ -35A - 12B &= 1 \end{aligned} \Rightarrow \begin{aligned} A &= -35/1369 \\ B &= -12/1369 \end{aligned}$$

This yields

$$\boxed{y_p = \frac{-35}{1369}\cos 2x - \frac{12}{1369}\sin 2x}$$

$\blacksquare$

<u>Ex</u>: Find <u>one</u> sol'n to

$$y'' + y' - 2y = 1 \cdot e^x \sin x$$

<u>Sol'n</u>: We "guess":

$$y = Ae^x \cos x + Be^x \sin x$$
$$= e^x (A\cos x + B\sin x)$$
$$y' = e^x((A+B)\cos x + (B-A)\sin x)$$
$$y'' = e^x(2B\cos x - 2A\sin x)$$

Now plug into ODE:

$$\underline{y'' + y' - 2y} \underset{\text{alg.}}{=} e^x((3B-A)\cos x + (-3A-B)\sin x)$$
$$\underset{\text{want}}{=} e^x \sin x$$

So we need: $3B - A = 0$, $-3A - B = 1$ $\Rightarrow$ $A = -3/10$, $B = -1/10$

This yields

$$\boxed{y_p = e^x\left(\frac{-3}{10}\cos x - \frac{1}{10}\sin x\right)}$$

$\blacksquare$

$\underline{Q}$: Can this approach ever $\underline{\text{fail}}$ to find y_p?

$\underline{Ex}$: Find $\underline{\text{one}}$ soln to

$$y'' - y = e^{-x}$$

$\underline{\text{Soln}}$: We "guess"

$$\begin{cases} y = Ae^{-x} \\ y' = -Ae^{-x} \\ y'' = Ae^{-x} \end{cases}$$

Now plug into ODE:

$$\underline{y'' - y} = Ae^{-x} - Ae^{-x} = 0$$

Not the same!

$$\text{want} \longrightarrow \; = e^{-x}$$

$\underline{\text{Problem}}$: Guess $y = Ae^{-x}$ solves the compl. eqn.!

Modification: Multiply orig. guess by x:

$$\begin{cases} y = Axe^{-x} \\ y' = (-Ax+A)e^{-x} \\ y'' = (Ax-2A)e^{-x} \end{cases}$$

Plug into ODE:

$$\underline{y''-y} = \underset{\text{alg.}}{\uparrow}{=} -2Ae^{-x} \underset{\text{want}}{\uparrow}{=} e^{-x}$$

Need: $-2A = 1 \Rightarrow A = -1/2$

This yields

$$\boxed{y_p = -\frac{1}{2}xe^{-x}}$$

∎

Moral: If any term of the "guess" is a sol'n of the compl. eqn., then mult. guess by x. Repeat if necessary.

Ex: Solve $y'' + 4y = x\cos 2x$.

Sol'n: 1. Solve the compl. eqn.

$$y'' + 4y = 0$$

$\Big\updownarrow$ char. eqn.

$$r^2 + 4 = 0 \Rightarrow r^2 = -4$$

$$\Rightarrow r = \pm 2i$$

$$\Rightarrow y_h = c_1\cos 2x + c_2 \sin 2x$$

2. Find a particular sol'n by "guessing"

$$y = (Ax+B)\cos 2x + (Cx+D)\sin 2x$$

Problematic terms

$\downarrow$ mult. by x

✓ $y = (Ax^2+Bx)\cos 2x + (Cx^2+Dx)\sin 2x$

Plugging into ODE we get

$$y_p'' + 4y_p = (8Cx + 2A + 4D)\cos 2x$$
$$+ (-8Ax - 4B + 2C)\sin 2x$$

alg. $= x \cos 2x$

want

Need: $8C = 1 \quad -8A = 0$

$2A + 4D = 0 \quad -4B + 2C = 0$

$\Rightarrow A = D = 0, \; C = \frac{1}{8}, \; B = \frac{1}{16}$

This yields

$$y_p = \frac{1}{16} x \cos 2x + \frac{1}{8} x^2 \sin 2x$$

3. The gen. sol'n is

$$\boxed{\begin{aligned} y &= y_p + y_h \\ &= \frac{1}{16} x \cos 2x + \frac{1}{8} x^2 \sin 2x \\ &\quad + c_1 \cos 2x + c_2 \sin 2x. \end{aligned}}$$

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