Infinite Series

An infinite series has the form:

$$a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{i=1}^{\infty} a_i = \sum a_i$$

Limits of Summation

Can be any variable.

(i.e. a series is a sum of a sequence)

Q: What does

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

mean?

A: We form the partial sums

$$S_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

$$\Big\downarrow \text{ let } n \to \infty$$

$$\boxed{\sum_{i=1}^{\infty} a_i = \boxed{\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i}}$$

When $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ exists we say that $\displaystyle\sum_{i=1}^{\infty} a_i$ converges. Otherwise it diverges. If $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} a_i = \infty$, we write $\displaystyle\sum_{i=1}^{\infty} a_i = \infty$.

Geometric Series:

$$1 + r + r^2 + r^3 + \cdots = \sum_{i=0}^{\infty} r^i$$

(Sum of geom. seq. $\{r^n\}_{n=0}^{\infty}$).

We have:

$$\begin{aligned} S_n &= 1 + \cancel{r} + \cancel{r^2} + \cancel{\ } \cdots + \cancel{r^n} \\ -\left(r S_n \right. &= \left. \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^n} + r^{n+1}\right) \end{aligned}$$

$$S_n(1-r) = 1 - r^{n+1} \quad r \neq 1$$

$$\Rightarrow \quad S_n = \frac{1-r^{n+1}}{1-r} = \underline{1 + r + \cdots + r^n}$$

$$(r=1) = n+1$$

So:

$$\sum_{i=0}^{\infty} r^i = \lim_{n\to\infty} S_n$$

r = 1

$$= \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} \quad \left(\text{or} \quad n+1\right)$$

Recall:

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & |r|<1 \\ \text{DNE}, & |r|>1 \\ \cancel{1, \quad r=1} \\ \text{DNE}, & r=-1 \end{cases}$$

Thus:

$$\boxed{1+r+r^2+\cdots = \sum_{i=1}^{\infty} r^i = \begin{cases} \dfrac{1}{1-r}, & |r|<1 \\ \text{DNE}, & |r|\geq 1 \end{cases}}$$

↳ Sum of Geom. Series

Ex:

$$0.333\ldots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots$$

$$= 3\left(\underbrace{\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots}\right)$$

Geom. Series, $r = \frac{1}{10}$,
starting w/ $i = 1$.

$$= 3\left(\underbrace{1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots}_{\text{Above}} + (-1)\right)$$

$$= 3\left(\frac{1}{1 - 1/10} - 1\right)$$

$$= 3\left(\frac{1}{9/10} - 1\right) = 3\left(\frac{10}{9} - 1\right)$$

$$= 3\left(\frac{10}{9} - \frac{9}{9}\right) = 3\left(\frac{1}{9}\right) = \frac{1}{3} \checkmark$$

$\blacksquare$

Properties of Series

If $\sum a_i$, $\sum b_i$ both converge:

1. $\left(\sum a_i\right) \pm \left(\sum b_i\right) = \sum (a_i \pm b_i)$

2. $c\left(\sum a_i\right) = \sum (c a_i)$

Theorem: If $\sum a_i$ converges, then $a_i \to 0$.

Proof: $s_n = a_1 + a_2 + \cdots + a_n \to \sum a_i$.

$$s_n - s_{n-1} = a_n$$

$$\downarrow n \to \infty$$

$$\underbrace{\left(\sum a_i\right) - \left(\sum a_i\right)}_{0} = \lim_{n\to\infty} a_n$$

∎

Infinite Series

An infinite series has the form:

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Limits of Summation

Index: can be any variable

(i.e. a series is a sum of a sequence)

Q: What does

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$$

mean?

A: Define the partial sums of $\sum a_i$ to be:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$s_n = a_1 + a_2 + \dots + a_n$$

$$\downarrow \quad \vdots \quad \downarrow \quad (\text{as } n \to \infty)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} s_n = \sum_{i=1}^{\infty} a_i$$

Def: We define

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \overbrace{\sum_{i=1}^{n} a_i}^{s_n},$$

provided the limit exists. In this case, we say $\sum_{i=1}^{\infty} a_i$ converges and call $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ its sum.

Warning: Given a series $\sum a_i$ there are two sequences that are relevant:

$$\{a_n\}, \quad \{S_n\}$$

seq. of indiv. terms

seq. of partial sums.

Geometric Series

$$1 + r + r^2 + r^3 + \cdots = \sum_{i=0}^{\infty} r^i = ?$$

By defn

$$\sum_{i=0}^{\infty} r^i = \lim_{n\to\infty} \sum_{i=0}^{n} r^i$$

$$= \lim_{n\to\infty} \boxed{\underbrace{(1 + r + r^2 + \cdots + r^n)}_{S_n}}$$

Notice:

$$S_n = 1 + \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^n}$$

$$-\left(rS_n = \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^n} + r^{n+1} \right)$$

$$S_n - rS_n = 1 - r^{n+1}$$

$$\begin{array}{c} \| \\ S_n(1-r) \end{array} \underset{r \neq 1}{\Rightarrow} S_n = \frac{1-r^{n+1}}{1-r}$$

So:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r}$$

$$= \begin{cases} \frac{1}{1-r}, & |r|<1 \\ \text{DNE}, & |r| \geq 1 \end{cases}$$

$$\boxed{\begin{array}{l} \underline{\text{Theorem}}: \text{The sum of a geometric series is} \\ 1 + r + r^2 + \cdots = \displaystyle\sum_{i=0}^{\infty} r^i = \begin{cases} \frac{1}{1-r}, & |r|<1 \\ \text{DNE}, & |r| \geq 1 \end{cases} \end{array}}$$

Properties of Series:

If $\sum a_i$, $\sum b_i$ **both** converge:

1. $\left(\sum a_i\right) \pm \left(\sum b_i\right) = \sum (a_i \pm b_i)$

2. $c\left(\sum a_i\right) = \sum (c a_i)$

Proof: These follow from the def'n of convergence of a series and the limit laws. ∎

Ex:

1. $0.333\ldots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$

$$= \boxed{\sum_{i=1}^{\infty} \frac{3}{10^i}} = 3 \sum_{i=1}^{\infty} \frac{1}{10^i}$$

$$= 3 \underbrace{\sum_{i=1}^{\infty} \left(\frac{1}{10}\right)^i}_{} = 3\left(\frac{1}{10} + \left(\frac{1}{10}\right)^2 + \cdots\right)$$

$\hookrightarrow$ Geom. series w/ $r = 1/10$

$$= 3\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots + (-1)\right)$$

$$= 3\left(\frac{1}{1 - 1/10} - 1\right)$$

$$= 3\left(\frac{1}{9/10} - 1\right)$$

$$= 3\left(\frac{10}{9} - 1\right) = 3 \cdot \frac{1}{9} = \frac{1}{3} \checkmark$$

A similar computation shows that

$$0.9999\ldots = 1. \qquad \blacksquare$$

2. $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$

$$= \frac{1}{2}\left(1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots\right)$$

Geom. series w/ $r = 1/2$

$$= \frac{1}{2} \cdot \frac{1}{1 - 1/2} = \frac{1}{2} \cdot \frac{1}{1/2}$$

$$= 1$$

$\boxtimes$