Divergence Test

Suppose $\sum a_n$ converges. Then

$$\lim_{n\to\infty} s_n = L$$

where

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

$$= \underbrace{a_1 + a_2 + \cdots + a_{n-1}}_{s_{n-1}} + a_n$$

$$\Rightarrow \quad s_n - s_{n-1} = a_n$$

$$\Rightarrow \quad \lim_{n\to\infty} (\underbrace{s_n}_{\downarrow L} - \underbrace{s_{n-1}}_{\downarrow L}) = \lim_{n\to\infty} a_n$$

$$0 = \lim_{n\to\infty} a_n$$

Theorem: If $\sum a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

$$\Updownarrow$$

$\boxed{\text{Divergence Test: If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum a_n \text{ diverges.}}$

Ex:

1. Since $\displaystyle\lim_{n\to\infty} \frac{2n^2+3}{n^2+1} = 2 \neq 0$,

   $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3}{n^2+1}$ diverges.

2. Since $\displaystyle\lim_{n\to\infty} \frac{n+1}{2n-3} = \frac{1}{2} \neq 0$,

   $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ diverges.

Warning: $\lim_{n\to\infty} a_n = 0$ **does not** guarantee $\sum a_n$ converges.

Ex: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. (**Harmonic Series:** $\sum_{n=1}^{\infty} \frac{1}{n}$)

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \frac{1}{\infty} = 0 \checkmark$$

However $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **diverges**:

$$S_n = \underbrace{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}}_{n \text{ terms}}$$

$$\geq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}$$

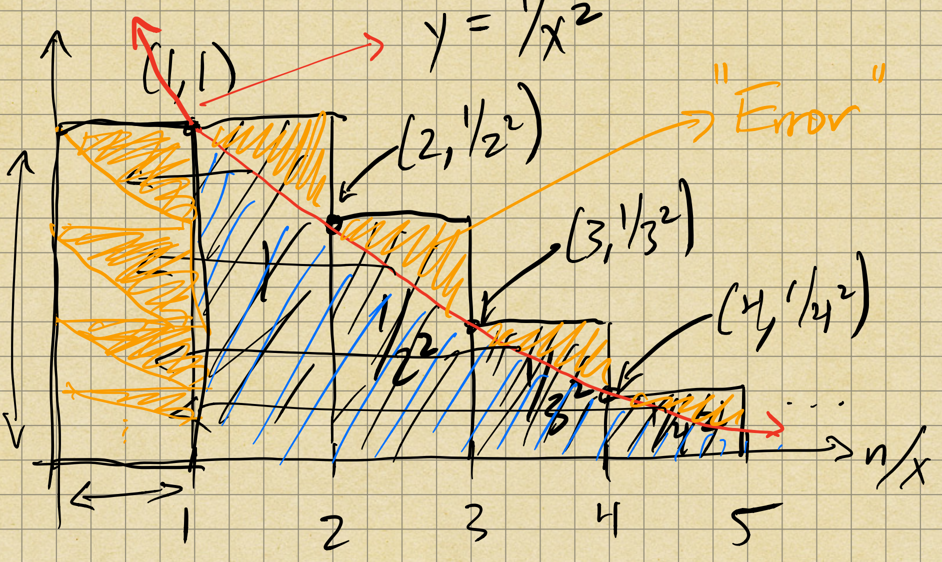
$$= \frac{n}{\sqrt{n}} = \sqrt{n}$$

$$\Rightarrow \quad S_n \geq \sqrt{n}$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges.}$$

The Integral Test

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$



Sum of box areas $= \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Idea: Compare box area to area under $y = \frac{1}{x^2}$. $\quad \displaystyle\int_1^{\infty} \frac{dx}{x^2}$

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^{t} \frac{dx}{x^2} = \lim_{t\to\infty} \left(\frac{-1}{x} \Big|_1^t \right)$$

$$= \lim_{t\to\infty}\left(\frac{-1}{t}+1\right) = 1$$

So $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges.

<u>Claim</u>: This implies $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$.

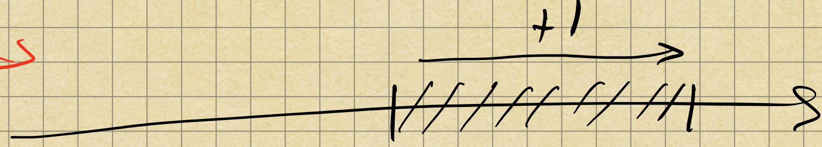
<u>Why</u>: $\displaystyle\int_1^\infty \frac{dx}{x^2} \approx \sum_{n=1}^\infty \frac{1}{n^2}$ (picture)

In fact:

Finite

$$\text{Error} = \sum_{n=1}^\infty \frac{1}{n^2} - \int_1^\infty \frac{dx}{x^2} \le 1$$

because we can "stack" indiv. errors into a 1×1 rectangle (see diagram).



$\displaystyle\int_1^\infty \frac{dx}{x^2}$ $\quad$ $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$ must live here

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges!

Moral: Convergence of $\int_1^{\infty} \frac{dx}{x^2}$ ensures convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Warning:

$$\underbrace{\int_1^{\infty} \frac{dx}{x^2}}_{1} \neq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\pi^2/6}$$

Integral Test: Suppose $f(x)$ is positive, continuous and decreasing. eventually is OK

Then

$$\sum_{n=1}^{\infty} f(n) \iff \int_1^{\infty} f(x)\,dx$$

converges $\qquad\qquad$ converges

"if and only if"

Ex:

1. Does $\sum_{n=1}^{\infty} \underbrace{\frac{n}{n^2+1}}_{f(n)}$ converge?

$f(n)$ where $f(x) = \frac{x}{x^2+1}$

Eventually pos. + dec.

Integral test $\Rightarrow$ series conv.

$$\Updownarrow$$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx \text{ conv.}$$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t \to \infty} \int_1^{t} \frac{x}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln(x^2+1)\Big|_1^t$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\ln(t^2+1) - \frac{1}{2}\ln 2\right)$$

$$= \infty$$

$$\Rightarrow \int_1^\infty \frac{x}{x^2+1}\,dx \ \text{diverges}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{n}{n^2+1} \ \text{diverges}}$$

Divergence Test

Suppose $\sum a_n$ converges to L:

$$\underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} a_n}_{S_n} = L.$$

Since

$$S_n = \overbrace{a_1 + a_2 + a_3 + \cdots + a_{n-1}}^{S_{n-1}} + a_n$$

$$\Rightarrow \quad a_n = S_n - S_{n-1}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (\underset{\downarrow}{\underset{L}{S_n}} - \underset{\downarrow}{\underset{L}{S_{n-1}}})$$

$$= 0$$

Theorem: If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. $\quad P \Rightarrow Q$

$$\Updownarrow \qquad\qquad \updownarrow$$

$$\neg Q \Rightarrow \neg P$$

Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Ex:

1. Since $\lim_{n\to\infty} \frac{2n^2-3}{3n^2+n} = \frac{2}{3} \neq 0$,

$$\sum_{n=1}^{\infty} \frac{2n^2-3}{3n^2+n} \text{ diverges}$$

2. Since $\lim_{n\to\infty} \frac{n+1}{2n-3} = \frac{1}{2} \neq 0$,

$$\sum_{n=1}^{\infty} \frac{n+1}{2n-3} \text{ diverges.}$$

3. Since $\lim_{n\to\infty} (-1)^n$ DNE,

$$\sum_{n=1}^{\infty} (-1)^n \text{ diverges.}$$

Warning: The converse of Div. Test is false, i.e.

$$\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Ex: Consider $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

We have:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \frac{1}{\infty} = 0.$$

But:

$$\boxed{S_n} = \underbrace{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}}_{n \text{ terms}}$$

$$\boxed{\geq} \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}$$

$$= \frac{n}{\sqrt{n}} = \boxed{\sqrt{n}}$$

$$\Rightarrow \quad \underline{S_n \geq \sqrt{n}} \xrightarrow{n\to\infty} \infty$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n = \infty$$

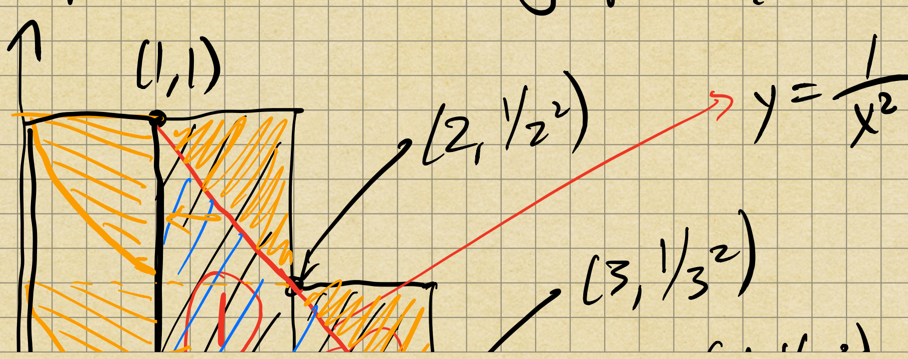
$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty \quad \text{(i.e. it \underline{diverges})}$$

<u>Moral</u>: Knowing $\lim_{n\to\infty} a_n = 0$ tells you <u>nothing</u> about convergence of $\sum_{n=1}^{\infty} a_n$.

Integral Test

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Let's represent the sum graphically:



Sum of boxed areas $= \sum_{n=1}^{\infty} \frac{1}{n^2}$.

Idea: Compare $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to $\int_1^{\infty} \frac{dx}{x^2}$.

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^{t} \frac{dx}{x^2} = \lim_{t\to\infty} \left(\frac{-1}{x} \Big|_1^t \right)$$

$$= \lim_{t\to\infty} \left(1 - \frac{1}{t} \right) = 1$$

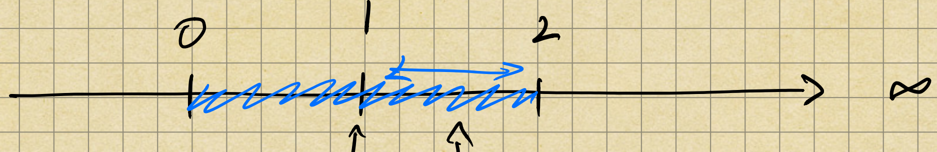
Diagram shows

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx \int_1^{\infty} \frac{dx}{x^2}$$

$$\text{Error} = \left| \sum_{n=1}^{\infty} \frac{1}{n^2} - \int_1^{\infty} \frac{dx}{x^2} \right|$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - \int_1^{\infty} \frac{dx}{x^2} \leq 1,$$

since "errors" can be stacked neatly in a 1×1 rectangle.



$\int_1^{\infty} \frac{dx}{x^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is in blue region.

__Q__: Can $\sum_{n=1}^{\infty} \frac{1}{n^2}$ diverge?

↳ Because terms are >0, either conv. or div. to ∞.

__A__: No! It __must__ be in the blue interval above!

Concl: $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}}$

↳ (In fact, converges to $\frac{\pi^2}{6}$.)

Integral Test: Suppose $f(x)$ is positive, continuous and decreasing. Then

"eventually"

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

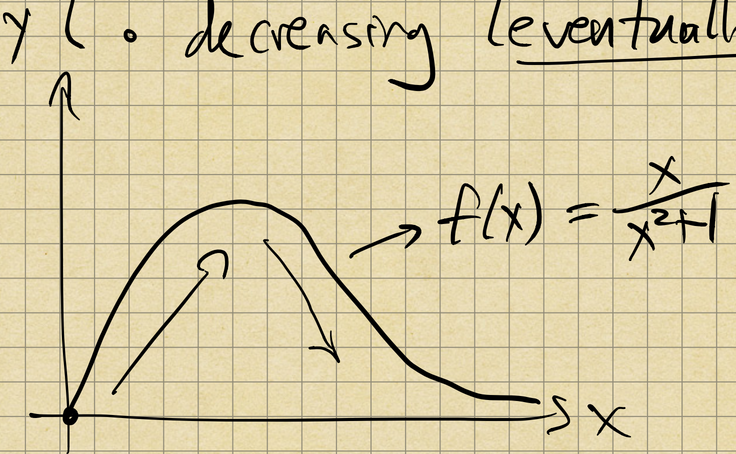
"if and only if"

Moral: To test convergence of $\sum_{n=1}^{\infty} a_n$ where $a_n = f(n)$, it suffices to compute $\int_1^{\infty} f(x)\,dx$.

<u>Ex</u>: 1. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converge?

Let $f(x) = \dfrac{x}{x^2+1}$. Then f is:

- positive (for $x > 0$) ✓
- continuous ✓
- decreasing (<u>eventually</u>) ✓



So we may apply Integral Test:

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{x}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2} \ln(x^2+1) \Big|_1^{t}$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\ln(t^2+1) - \frac{1}{2}\ln 2\right)$$

$$= \infty$$

Concl.:

$$\int_1^\infty \frac{x}{x^2+1}\,dx \Longrightarrow \boxed{\sum_{n=1}^\infty \frac{n}{n^2+1} \text{ diverges}}$$

diverges

Warning: Integral test does not guarantee that

$$\cancel{\sum_{n=1}^\infty f(n) = \int_1^\infty f(x)\,dx}.$$

e.g. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ but $\displaystyle\int_1^\infty \frac{dx}{x} = 1$.

Not equal!

2. Does $\displaystyle\sum_{n=2}^\infty \frac{1}{n\cdot \ln n}$ converge?

$$f(x) = \frac{1}{x \cdot \ln x} \quad \text{is :}$$

- positive (for $x > 1$)
- continuous (for $x > 1$)
- decreasing (eventually)

So we can apply Integral Test:

$$\int_2^\infty \frac{dx}{x \cdot \ln x} = \text{"} \ln(\ln x) \Big|_1^\infty \text{"}$$

$$= \infty \Rightarrow \text{integral div.}$$

$$\Rightarrow \boxed{\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ div.}}$$