p-series

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

When does this converge?

↳ For what p?

↑ Riemann zeta function

Use integral test:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx$$

↑ Continuous, Decreasing, Positive

$p \neq 1$

$$= \lim_{t\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^t = \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right)$$

$$= \begin{cases} \infty, & 1-p > 0 \iff p \leq 1 \\ \frac{1}{p-1}, & 1-p < 0 \iff p > 1 \end{cases}$$

When $p=1$:

$$\int_1^\infty x^{-1}\,dx = \int_1^\infty \frac{dx}{x} = \ln x \Big|_1^\infty = \infty$$

Conclusion: The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges iff $p > 1$.

Ex:

1. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$p = 1 \leq 1$

diverges.

2. The series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$$

$p = 1/2 \leq 1$

diverges (which we already knew).

3. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \quad \left(= \frac{\pi^4}{90} \right)$$

$p = 4 > 1$

<u>converges</u> (although we don't have a way to determine it's value).

Comparison Test

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)}$.

We apply the integral test:

$$\int_1^\infty \frac{dx}{x(x+1)(x+2)(x+3)}$$

(PFD) $$= \int_1^\infty \frac{1/6}{x} - \frac{1/2}{x+1} + \frac{1/2}{x+2} - \frac{1/6}{x+3}\,dx$$

$$= \lim_{t\to\infty}\left(\frac{1}{6}\ln x - \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x+2) - \frac{1}{6}\ln(x+3)\Big|_1^t\right)$$

$$= \lim_{t\to\infty}\left(\frac{1}{6}\ln t - \frac{1}{2}\ln(t+1) + \frac{1}{2}\ln(t+2) - \frac{1}{6}\ln(t+3) + \underline{\frac{1}{2}\ln 2 - \frac{1}{2}\ln 3 + \frac{1}{6}\ln 4}\right)$$

$$= \lim_{t\to\infty}\left(\frac{1}{6}\ln t - \frac{3}{6}\ln(t+1) + \frac{3}{6}\ln(t+2)\right.$$

$$- \frac{1}{6}\ln(t+3) + \cdots \Big)$$

$$= \lim_{t\to\infty}\left(\frac{1}{6}\ln t - \frac{1}{6}\ln(t+1)^3 + \frac{1}{6}\ln(t+2)^3 - \frac{1}{6}\ln(t+3) + \cdots\right)$$

$$= \lim_{t\to\infty}\left(\frac{1}{6}\ln\underbrace{\left(\frac{t(t+2)^3}{(t+1)^3(t+3)}\right)}_{=\frac{t^4+\cdots}{t^4+\cdots}\to 1} + \cdots\right)$$

$$= \frac{1}{6}\overset{0}{\cancel{\ln 1}} + \cdots < \infty$$

$\Rightarrow$ integral converges

$\Rightarrow$ $\boxed{\text{series converges}}$

<u>Better Idea</u>: For large n,

$$\frac{1}{n(n+1)(n+2)(n+3)} < \frac{1}{n^4}$$

This suggests that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)} \iff \boxed{\sum_{n=1}^{\infty} \frac{1}{n^4}}$$

converges $\quad$ converges ✓

p-series w/ $p = 4 > 1$

(Limit) Comparison Test:

Suppose $a_n, b_n > 0$ for $n \geq 1$,

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c.$$

If $c \neq 0, \infty$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{iff} \quad \sum_{n=1}^{\infty} b_n$$

converges $\quad$ Converges

Ex:

1. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$ converge?

$$\approx \frac{n^2}{3n^4} = \frac{1}{3n^2}$$

So we compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{\left(\frac{n^2-1}{3n^4+1}\right)}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{n^2(n^2-1)}{3n^4+1} = \lim_{n\to\infty} \frac{n^4-n^2}{3n^4+1} = \frac{1}{3}$$

Since $\frac{1}{3} \neq 0, \infty$, our series converges iff $\underbrace{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}}_{\text{p-series of } p=2>1}$ ✓

$$\boxed{\text{Yes, the orig. series converges.}}$$

2. Does $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ converge?

$$\approx \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$

So we limit compare w/ $\boxed{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}}$: $\rightarrow$ p-series, $p = \frac{1}{2} \leq 1$ $\Rightarrow$ diverges

$$\lim_{n\to\infty} \frac{\left(\frac{\sqrt{n}}{n-1}\right)}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{n}{n-1}$$

$$= 1 \neq 0, \infty$$

Thus, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n-1}$ **does not converge,**

since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

3. Does $\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$ converge?

$$\approx \frac{4^n}{3^n} = \left(\frac{4}{3}\right)^n$$

So we limit compare w/ $\sum \left(\frac{4}{3}\right)^n$,

$n=1$

which is a <u>geometric</u> series w/

$r = 4/3 > 1$, and hence divergent.

$$\lim_{n\to\infty} \frac{\frac{1+4^n}{1+3^n}}{(4/3)^n}$$

$$= \lim_{n\to\infty} \frac{3^n(1+4^n)}{4^n(1+3^n)}$$

$$= \lim_{n\to\infty} \frac{\cancel{12^n}(4^{-n}+1)}{\cancel{12^n}(3^{-n}+1)}$$

$4^{-n} \to 0$, $3^{-n} \to 0$

$$= 1 \neq 0, \infty$$

$$\Rightarrow \boxed{\text{Orig. series is divergent}}$$

<u>p-series</u> For what $p > 0$ does

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converge?}$$

If $p \leq 0$, $\frac{1}{n^p} \not\to 0$, so series diverges.

↳ <u>Riemann zeta function</u>

We use the integral test:

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx \xrightarrow{p \neq 1} \frac{x^{1-p}}{1-p}\Bigg|_1^\infty$$

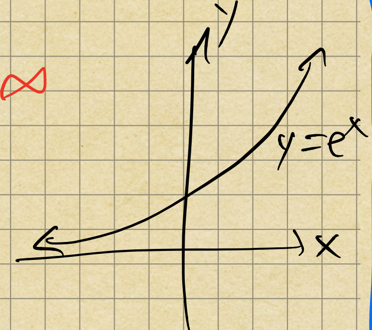
↳ Positive, cont., decreasing

$$= \lim_{t\to\infty}\left(\frac{\boxed{t^{1-p}}}{1-p} - \frac{1}{1-p}\right)$$

$$= \begin{cases} \infty, & 1-p>0 \iff 1>p \\ \dfrac{1}{p-1}, & 1-p<0 \iff 1<p \end{cases}$$

$$\lim_{t\to\infty} t^A = \lim_{t\to\infty}\left(e^{\ln t}\right)^A \qquad \infty$$

$$= \lim_{t\to\infty} e^{A\ln t}$$



$$= \begin{cases} \infty, & A>0 \\ 1, & A=0 \end{cases}$$

$$\begin{cases} \;\; 1 \; , & A > 0 \\ \;\; 0 \; , & A < 0 \end{cases}$$

When $p=1$ we have

$$\int_1^\infty \frac{dx}{x} = \ln x \Big|_1^\infty = \infty$$

$$\boxed{\begin{array}{l} \underline{\text{Conclusion}}: \text{ The } p\text{-series} \\ \displaystyle\qquad\qquad \sum_{n=1}^\infty \frac{1}{n^p} \\ \text{converges iff } p > 1. \end{array}}$$

<u>Ex</u>:

1. The <u>harmonic series</u>

$$\sum_{n=1}^\infty \frac{1}{n} = \infty$$

This is equivalent to the fact that there are infinitely many prime #'s.

↳ p-series, $p = 1 \leq 1$

diverges.

Remark: If $a_n > 0$, then

$$\sum_{n=1}^{\infty} a_n$$

is either converges or diverges to ∞.

2. $$\sum_{n=3}^{\infty} n^{-0.9999} = \sum_{n=3}^{\infty} \frac{1}{n^{0.9999}} = \infty$$

↳ p-series w/ $p = 0.9999 \leq 1$

which we already know.

3. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

↳ p-series w/ $p = 4 > 1$

<u>converges</u>.

Comparison Test

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)}$.

We may apply the integral test:

$$\int_1^{\infty} \frac{dx}{x(x+1)(x+2)(x+3)}$$

$$(PFD) = \int_1^{\infty} \frac{1/6}{x} - \frac{1/2}{x+1} + \frac{1/2}{x+2} - \frac{1/6}{x+3}\, dx$$

$$= \frac{1}{6}\ln x - \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x+2) - \frac{1}{6}\ln(x+3)\Big|_1^{\infty}$$

$$= \lim_{t\to\infty}\left(\frac{1}{6}\ln\left(\frac{t(t+2)^3}{(t+3)(t+1)^3}\right) + \text{constant}\right)$$

$$= \frac{1}{6}\ln\left(\lim_{t\to\infty}\frac{t(t+2)^3}{(t+3)(t+1)^3}\right) + \text{constant}$$

$$= \frac{1}{6} \ln\left(\lim_{t \to \infty} \frac{t^4 + \cdots}{t^4 + \cdots} \right) + \text{constant}$$

$$= \frac{1}{6} \underbrace{\cancel{\ln(1)}}_{0} + \text{const.} < \infty$$

$\Rightarrow$ integral converges

$\Rightarrow$ $\boxed{\text{series converges}}$ **Whew!**

<u>Better Idea:</u>

$$\frac{1}{n(n+1)(n+2)(n+3)} \underset{\substack{\uparrow \\ n \to \infty}}{\approx} \frac{1}{n^4}$$

So the terms in

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)}$$

eventually "look like" the terms

in

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$ which is convergent.

p-series w/ $p = 4 > 1$

So we expect the original series to converge, too.

(Limit) Comparison Test

If $a_n, b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0, \infty$$

then:

$\sum a_n$ converges iff $\sum b_n$ converges

Ex:

1. Consider $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{n(n+1)(n+2)(n+3)}}$

$\approx \frac{1}{n^4}$

We (limit) compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$:

$$\lim_{n\to\infty} \left(\frac{\frac{1}{n(n+1)(n+2)(n+3)}}{1/n^4} \right)$$

$$= \lim_{n\to\infty} \frac{n^4}{n(n+1)(n+2)(n+3)}$$

$$= \lim_{n\to\infty} \frac{n^4}{n^4 + \cdots} = \underbrace{1 \neq 0, \infty}$$

limit comparison worked!

So $\boxed{\text{orig. series converges}}$ since

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges.

2. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$ converge?

$$\approx \frac{4^n}{3^n} = \left(\frac{4}{3}\right)^n$$

We limit compare to $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$:

$$\lim_{n\to\infty} \frac{\left(\frac{1+4^n}{1+3^n}\right)}{(4/3)^n}$$

$$= \lim_{n\to\infty} \frac{3^n(1+4^n)}{4^n(1+3^n)}$$

$$= \lim_{n\to\infty} \frac{\cancel{3^n}\,\cancel{4^n}(4^{-n}+1)}{\cancel{4^n}\,\cancel{3^n}(3^{-n}+1)}$$

$$= \lim_{n\to\infty} \frac{4^{-n}+1}{3^{-n}+1} = \frac{0+1}{0+1} = 1 \neq 0, \infty$$

So both series are either conv.

or div. Since

$$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$$

is a <u>geometric</u> series w/

$$r = 4/3 > 1,$$

it diverges. So $\boxed{\text{orig. series diverges.}}$