Alternating Series

↳ Has the form

$$\sum_{n=1}^{\infty} (-1)^n a_n = \underset{-}{\underline{-a_1}} \underset{+}{\underline{+a_2}} \underset{-}{\underline{-a_3}} \underset{+}{\underline{+a_4}} - \cdots$$

- OR -

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \underset{+}{\underline{a_1}} \underset{-}{\underline{-a_2}} \underset{+}{\underline{+a_3}} \underset{-}{\underline{-a_4}} + \cdots$$

where $a_n \geq 0$ for all n.

Ex : $n=1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \ldots$

1. $-\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \cdots$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n n}{n+2} \text{ is alternating}$$

2. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad \underline{\text{is}} \text{ alternating}$$

↳ alternating harmonic series

3. $$\sum_{n=1}^{\infty} \boxed{\underline{(-1)^{n+1}}}\ \boxed{\sin\left(\frac{n\pi}{2}\right)}$$

↳ $\underline{\text{not}}$ always ≥ 0

$$\overset{1}{=} \sin(\pi/2) \overset{0}{-} \sin(2\pi/2) + \sin(3\pi/2) \; -1$$

$$\overset{0}{} - \sin(4\pi/2) + \sin(5\pi/2) \overset{+1}{-} \cdots$$

is $\underline{\text{not}}$ alternating!

<u>Alternating Series Test</u>: Suppose:

1. $a_n \geq 0$ → Shorthand: $a_n \searrow 0$.

$\{a_n\}$ 2. $\boxed{a_1 \geq a_2 \geq a_3 \geq \cdots \longrightarrow 0}$

decreases to 0

$\hookrightarrow a_n$ are <u>decreasing</u>

3. $\lim_{n \to 0} a_n = 0$

or $(-1)^n$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ <u>converges</u>.

<u>Ex</u>:

1. Since $\{1/n\}$ is positive and <u>decreases to 0</u>:

$$\left(\checkmark \; 1 \geq \frac{1}{2} \geq \frac{1}{3} \geq \frac{1}{4} \geq \cdots \to 0 \; \checkmark \right)$$

$$AST \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges}}$$

"Alternating Series Test"

<u>Remarks</u>:

1. Recall that $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u> w/o $\pm$ signs!

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

2. Since $\left\{ \dfrac{1}{\ln n} \right\}_{n=2}^{\infty}$ is positive

and decreases to 0, so

$$\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \quad \text{converges}}$$

3. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^4+2}$.

$\dfrac{n^2}{n^4+2}$ is positive and

$$\lim_{n\to\infty} \frac{n^2}{n^4+2} = 0 \quad (4>2)$$

Does $\left\{ \underbrace{\dfrac{n^2}{n^4+2}}_{f(n)} \right\}$ decrease?

$$f'(n) = \frac{(n^4+2)(2n) - n^2(4n^3)}{(n^4+2)^2}$$

$$= \frac{2n^5 + 4n - 4n^5}{(n^4+2)^2}$$

$$= \frac{-2n^5 + 4n}{(n^4+2)^2} < 0$$

always pos.

since eventually $4n < 2n^5$

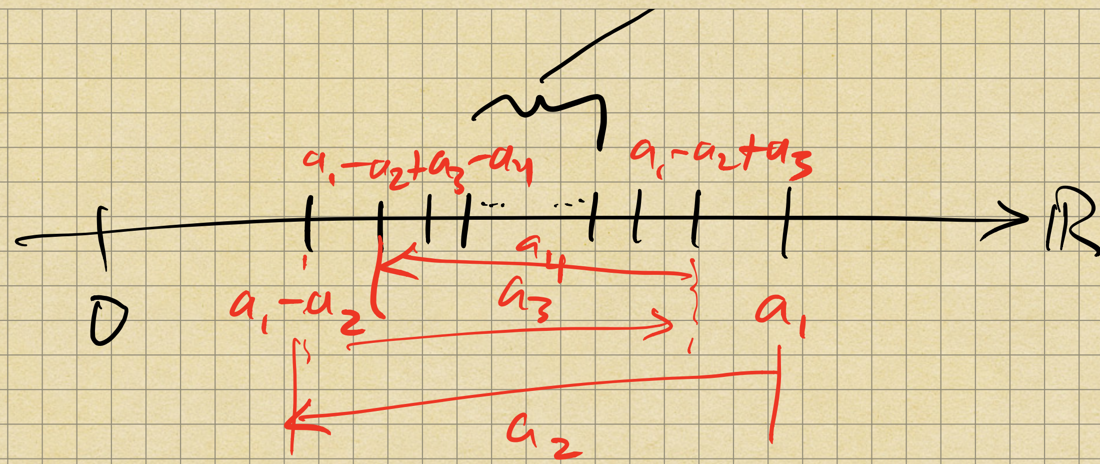
$\Rightarrow$ eventually $\dfrac{n^2}{n^4+2}$ decreases.

So AST $\Rightarrow$ $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^4+2} \text{ converges}}$

Why does AST work?

We have $a_1 \geq a_2 \geq a_3 \geq a_4 \cdots \longrightarrow 0$

Consider:



$$a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

So series converges!

Alternating Series

↳ Have the form:

$$\sum_{n=1}^{\infty} (-1)^n a_n = \underbrace{-a_1}_{-} \underbrace{+a_2}_{+} \underbrace{-a_3}_{-} \underbrace{+a_4}_{+} - \cdots$$

-OR-

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \underbrace{a_1}_{+} \underbrace{-a_2}_{-} \underbrace{+a_3}_{+} \underbrace{-a_4}_{-} + \cdots$$

<u>where</u> $a_n \geq 0$ for all n.

Ex:

1. $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+2} \text{ is alternating.}$$

$\hookrightarrow a_n \geq 0$

2. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \text{ is alternating.}$$

$\hookrightarrow a_n = \frac{1}{n} \geq 0$

$\hookrightarrow$ Alternating Harmonic Series

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{n\pi}{2}\right)$

$= 0 \geq 0$ ✓

$= \sin(\pi/2) - \sin(2\pi/2)$

$+\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{4\pi}{2}\right) + \cdots$

-1

is ~~not~~ alternating!

-1

$\cdots + \sin\left(\frac{7\pi}{2}\right)$

Why? $a_n = \sin\left(\frac{n\pi}{2}\right)$ is not ≥ 0 for all n! In fact,

$$-1 = \sin\left(\frac{3\pi}{2}\right) = \sin\left(\frac{7\pi}{2}\right) = \sin\left(\frac{11\pi}{2}\right) = \cdots$$

Alternating Series Test (AST): Suppose:

$\{a_n\}$ decreases to 0

1. $a_n \geq 0$
2. $\boxed{a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \rightarrow 0}$
3. $\lim_{n\to\infty} a_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Can also be n.

Ex:

1. Since $\{1/n\}_{n=1}^{\infty}$ decreases to 0,

AST $\Rightarrow$ $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges.}}$

$\downarrow$ Remove signs.

Remarks:

- Note that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

- Using power series one can show

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

2. Since $\{1/\ln n\}_{n=2}^{\infty}$ decreases to 0

$\left("\frac{1}{\ln n} \searrow 0" \right)$, AST $\Rightarrow$

$$\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \text{ converges.}}$$

3. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^4+2}$.

- $\dfrac{n^2}{n^4+2} \geq 0$ for all n ✓

- $\displaystyle\lim_{n\to\infty} \frac{n^2}{n^4+2} = 0$ ✓

  (deg. denom. > deg. numer.)

- $\underbrace{\dfrac{n^2}{n^4+2}}_{f(n)}$ is decreasing... ?

  $\Updownarrow$

  $f'(n) < 0$

$$f'(n) = \frac{(n^4+2)(2n) - n^2(4n^3)}{(n^4+2)^2}$$

$$= \frac{4n - 2n^5}{(n^4+2)^2} = \frac{\underbrace{-2n}_{\text{always } -}\ \underbrace{(n^4-2)}_{+ \text{ for } n>1}}{\underbrace{(n^4+2)^2}_{\text{always } +}}$$

< 0 if $n > 1$

$\Rightarrow$ $f(n)$ is eventually decreasing ✓

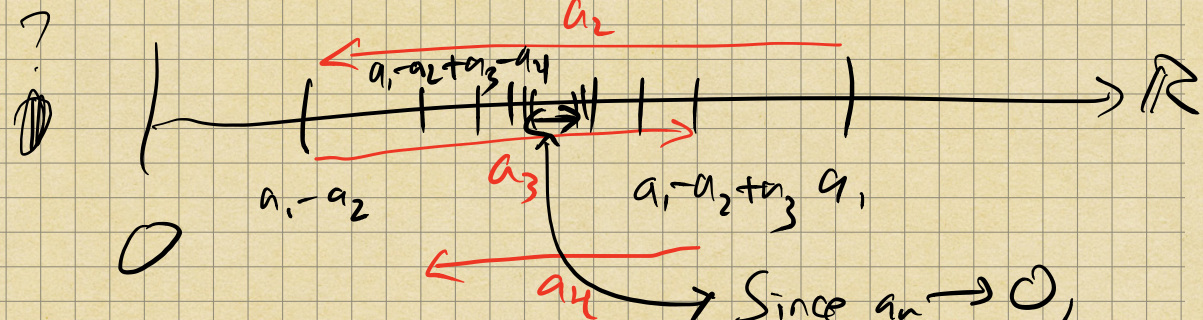
Concl: $\frac{n^2}{n^4+2} \searrow 0$ (eventually)

AST $\Rightarrow$ $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^4+2} \text{ converges}}$

Why does AST work?

Suppose $a_n \searrow 0$

Consider $a_1 - a_2 + a_3 - a_4 + a_5 \cdots + (-1)^{n+1} a_n$:



Since $a_n \to 0$, the "gap" closes

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ conv.}$$