Absolute Convergence

Let $\{a_n\}$ be a sequence. For any n:

$$0 \le \underline{a_n + |a_n|} \le |a_n| + |a_n|$$
$$= \underline{2\,|a_n|}$$

What if $\boxed{\sum |a_n| \text{ converges}}$?

$\Rightarrow$ $2 \sum |a_n|$ converges

$\Rightarrow$ $\boxed{\sum (a_n + |a_n|) \text{ converges}}$

by (direct) comparison

$\Rightarrow$ $\sum (a_n + |a_n|) - \sum |a_n|$ converges

$$\| $$

$$\sum a_n = \sum (a_n + \cancel{|a_n|} - \cancel{|a_n|})$$

Theorem:

$$\sum |a_n| \text{ converges} \Rightarrow \sum a_n \text{ converges}$$

Def: **absolute convergence**

Remark: If $\sum |a_n|$ <u>diverges</u>, then $\sum a_n$ may or <u>may not</u> converge.

→ **Conditional convergence**

Ex:

1. $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges by <u>AST</u> $\left(\frac{1}{n} \searrow 0\right)$

But:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n^1} \quad \text{diverges}$$

$\hookrightarrow$ p-series, $p=1\leq 1$

So $\boxed{\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \text{ converges conditionally}}$

2. $$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \text{converges}$$

$\hookrightarrow$ p-series, $p=2>1$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2} \text{ is absolutely convergent}}$$

3. $$\sum_{n=1}^{\infty}\frac{(-1)^{n}n}{n+2} \quad \text{is divergent by div. test}$$

$\hookrightarrow$ Do not $\to 0$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{n}{n+2} \quad \text{is divergent, too.}$$

Possible convergence "arrangements":

$\sum a_n$		$\sum \lvert a_n \rvert$		
Conv.	$\Leftarrow$	Conv.	$\rightarrow$	Abs. conv.
Conv.		Div.	$\rightarrow$	Cond. conv.
~~Div.~~		~~Conv.~~		
Div.	$\Rightarrow$	Div.	$\rightarrow$	Div.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ conv. or div. ?

Abs. value

$$\sum_{n=1}^{\infty} \left| \frac{\sin 4n}{4^n} \right| \le \sum_{n=1}^{\infty} \frac{1}{4^n} = \boxed{\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n}$$

comparison

↳ Geom. w/ $r = 1/4$

$\Rightarrow$ convergent

$\Rightarrow$ convergent

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n} \text{ is abs. conv.}}$$

General Strategy for Testing Conv. of $\sum a_n$:

- Test whether $a_n \to 0$.
- Test whether $\sum |a_n|$ conv.
- Test whether $\sum a_n$ conv. (use AST, for instance)

The Ratio Test: Suppose

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \text{exists.}$$

1. If $L < 1$, $\sum a_n$ is abs. conv.

2. If $L > 1$, $\sum a_n$ div.

3. If $L = 1$, ~~test~~ is inconclusive.

The Root Test: Suppose

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \quad \text{exists}$$

1. If $L < 1$, $\sum a_n$ conv. abs.

2. If $L > 1$, $\sum a_n$ div.

3. If $L = 1$, no info.

Ex:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ conv. or div.

$a_n = \dfrac{(-2)^n}{n!}$

Suggests we use Ratio Test

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}/(n+1)!}{(-2)^n/n!}\right|$$

$$= \lim_{n\to\infty}\frac{2^{\cancel{n+1}}\,\cancel{n!}}{\cancel{2^n}\,(n+1)!} = \lim_{n\to\infty}\frac{2}{n+1} = 0$$

$(n+1)! = (n+1)\cdot\cancel{n!}$

Since $0<1$, the orig. series

$$\boxed{\sum_{n=1}^{\infty}\frac{(-2)^n}{n!} \text{ conv. abs.}}$$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{5^n}$ conv. or div.?

Root Test:

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^{n-1}n}{5^n}\right|} = \lim_{n\to\infty}\sqrt[n]{\frac{n}{5^n}}$$

$e^{\frac{\ln n}{n}}$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{5} = \frac{1}{5} < 1$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n} \text{ conv. abs.}}$$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{2^{n+1}/(n+1)^{100}}{2^n/n^{100}} \right|$$

$$= \lim_{n\to\infty} \frac{2^{n+1} \; n^{100}}{2^n \; (n+1)^{100}}$$

$$= \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^{100} = 2\cdot 1^{100} = 2 > 1$$

So $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{100}} \text{ diverges}}$

Absolute Convergence

Suppose we are given $\sum a_n$ ...

$$0 \le \boxed{a_n + |a_n|} \le |a_n| + |a_n| = \boxed{2|a_n|}$$

Suppose we know $\sum |a_n|$ converges :

$\Rightarrow 2\sum |a_n|$ converges

Direct Comp. $\Rightarrow \sum (a_n + |a_n|)$ converges

$$\sum (a_n + |a_n|) - \sum |a_n| \quad \text{converges}$$

$$= \sum (a_n + \cancel{|a_n|} - \cancel{|a_n|})$$

$$= \sum a_n$$

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}: \\ \displaystyle \underset{\text{conv.}}{\sum |a_n|} \Rightarrow \underset{\text{conv.}}{\sum a_n}\end{array}}$$

<u>Absolute convergence</u>

<u>Remark</u>: If $\sum |a_n|$ <u>diverges</u>,

$\sum a_n$ <u>may</u> or may not converge.

<u>conditional convergence</u>

<u>Ex</u>: 1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by <u>AST</u>

$(1/n \searrow 0)$

<u>However</u>, $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$

Harmonic series

diverges.

So $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ is conditionally conv.}}$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

2. $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

p-series, $p=2$

So $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \text{ is abs. conv.}}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+2}$ $\boxed{\text{diverges by Div. Test}}$

Conv. to 0? No!

Possible Convergence "Arrangements"

$\sum a_n$	$\sum \lvert a_n \rvert$	
Conv. $\Leftarrow$	Conv.	Abs. Conv.
Conv.	Div.	Cond. Conv.
~~Div.~~	~~Conv.~~	
Div. $\Rightarrow$	Div.	Divergence

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ conv. or div.?

Test for abs. conv.:

$$\underbrace{\sum_{n=1}^{\infty} \left| \frac{\sin(4n)}{4^n} \right|}_{\text{Only non-neg. terms.}} \leq \sum_{n=1}^{\infty} \frac{1}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$$

$\leq$ ↳ Comparison

Geom. series $r = 1/4$, $\uparrow$ $|r| < 1$

↳ Conv.

$\Rightarrow$) converges

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n} \text{ converges absolutely}}$$

General Strategy for Testing Conv. of $\sum a_n$

1. Does $a_n \to 0$?

Yes

2. Does $\sum |a_n|$ conv. ?

No

3. Does $\sum a_n$ conv. ?

$\hookrightarrow$ Prob. need AST

We get two "new" tests for abs. conv.

Ratio Test: Suppose

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \quad \text{exists}$$

(∞ is OK)

1. If $L<1$, $\sum a_n$ conv. abs.

2. If $L>1$, $\sum a_n$ diverges

3. If $L=1$, test is inconclusive.

The Root Test: Suppose

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \quad \text{exists } (\infty \text{ is OK})$$

1. If $L<1$, $\sum a_n$ is abs. conv.

2. If $L>1$, $\sum a_n$ div.

3. If $L=1$, no info.

Ex:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1}/(n+1)!}{(-2)^n/n!} \right| = \lim_{n\to\infty} \frac{2^{n+1} \cdot n!}{2^n \cdot (n+1)!}$$

$\hookrightarrow (n+1)\cdot n!$

$$= \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{(-2)^n}{n!} \text{ conv. abs.}}$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n}$

Root Test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\left| \frac{(-1)^{n-1} n}{5^n} \right|}$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{n}{5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{5^n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{n}}{5} = \frac{1}{5} < 1$$

$$= e^{\frac{\ln n}{n}}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n} \text{ is abs. conv.}}$$

~~"The series is abs. conv."~~

"The given series is abs. conv."

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$ conv. or div.

<u>Ratio Test:</u>

$$\lim_{n\to\infty} \left| \frac{2^{n+1}/(n+1)^{100}}{2^n/n^{100}} \right|$$

$$= \lim_{n \to \infty} \frac{2 n^{100}}{(n+1)^{100}}$$

$$= \lim_{n \to \infty} \underline{\underline{2}} \left(\frac{n}{n+1} \right)^{200}$$

$$= 2 \cdot 1^{100} = 2 > 1$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{2^n}{n^{100}} \text{ diverges}}$$