Exam 3

When: Monday, 5/3

What: Method of Undetermined Coeff. (ODEs)

Sequences / Series

- Convergence
- Conv. Tests for Series
- Abs. / Cond. Conv.
- Power Series: def'n, radius of conv.

Q&A: In class, Monday

#13:

$$\sum_{n=3}^{\infty} \cancel{6}\left(1+\frac{1}{n}\right)^{n^2}$$

$$\lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2/n}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e > 1$$

Take ln of both sides first.

$\Rightarrow \boxed{\text{series div.}}$

#3: $$6\sum_{n=1}^{\infty} \underbrace{\frac{1+\sin(n)}{n^4}}_{} \le \cancel{6}\sum_{n=1}^{\infty} \frac{\cancel{2}1}{n^4} \rightarrow \text{Conv.}$$

$p = 4 > 1$

$$0 \le \frac{1+\sin(n)}{n^4} \le \frac{2}{n^4}$$

Direct Comparison

$$\boxed{\text{(Abs.) Conv.}}$$

Power Series

Series as Functions:

Ex: 1. The geom. series:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$|r| < 1$

2. The p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \zeta(p) \quad \text{converges for } p>1$$

P.S.

Def: A power series has the form:

coeffs.

$$\sum_{n=\boxed{0}}^{\boxed{\infty}} c_n (x-a)^n \Big\} \text{ Powers of } x-a$$

center

variable

Ex: 1. The geom. series

$$\sum_{n=0}^{\infty} x^n$$

is a P.S. w/ center $a=0$ and coeffs. $c_n = 1$. This conv. iff $|x|<1$.

2. For what x does $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

$\hookrightarrow$ $0! = 1$, $x^0 = 1$

(P.S. w/ center $a = 0$ and coeffs. $c_n = \frac{1}{n!}$)

<u>Ratio Test:</u>

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} \frac{n! \cdot |x|^{n+1}}{\underbrace{(n+1)!}_{=(n+1)\cdot n!} \cdot |x|^n}$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

for <u>any</u> x.

$$\Rightarrow \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ converges abs. for all real } x!}$$

3. Same question for $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n(-1)^{n+1}}{n}$

(P.S. w/ center $a = 1$ and coeffs. $c_n = \frac{(-1)^{n+1}}{n}$)

Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-1)^n(-1)^{n+1}}{n}\right|}$$

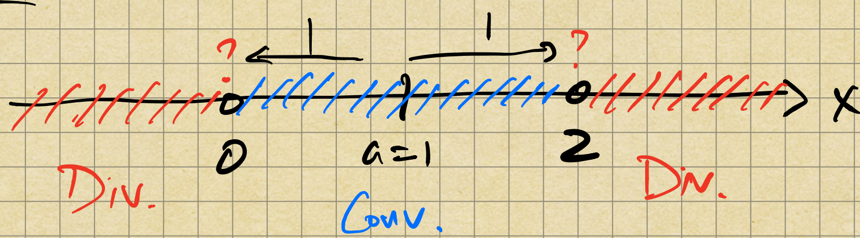
$$= \lim_{n\to\infty} \sqrt[n]{\frac{|x-1|^n}{n}} = \lim_{n\to\infty} \frac{|x-1|}{\sqrt[n]{n}}$$

$$= |x-1|$$

Conclusions:

1. $|x-1| < 1 \Rightarrow$ series conv. abs.
2. $|x-1| > 1 \Rightarrow$ series div.
3. $|x-1| = 1 \Rightarrow$ no info.

Picture:



$x=0$: Series is

$$\sum_{n=1}^{\infty} \frac{(0-1)^n(-1)^{n+1}}{n} = \sum_{n=1} \frac{-1}{n}$$

which is $\underline{\text{divergent}}$.

p-series, $p=1$

$\underline{x=2}$: Series is

$$\sum_{n=1}^{\infty} \frac{(2-1)^n(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

which is $\underline{\text{conv.}}$.

(AST, $1/n \searrow 0$)

So the domain (of convergence) is

$$\boxed{(0,2]}$$

In general:

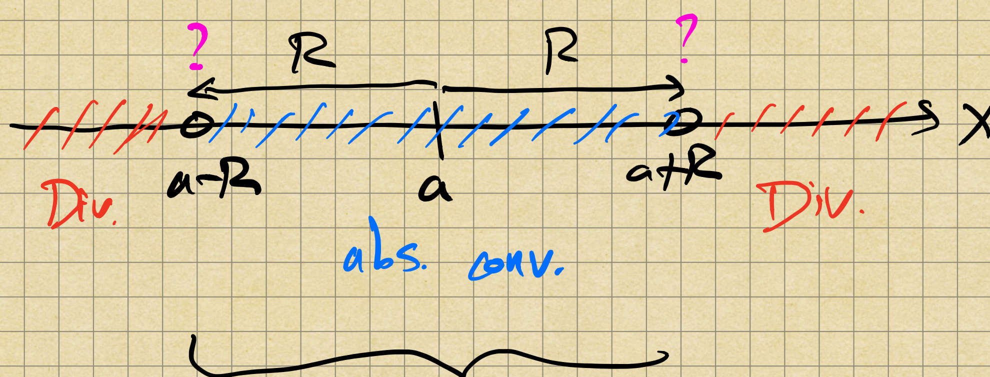
$\underline{\text{Theorem}}$: Given $\sum_{n=0}^{\infty} c_n (x-a)^n$, there is a number $0 \le R \le \infty$ (the $\underline{\text{radius of convergence}}$) so that:

1. The series conv. abs. for $|x-a| < R$

2. The series div. for $|x-a| > R$

[<u>Note</u>: No general info. on what happens when $|x-a| = R$]

<u>Picture</u>:



<u>Interval of Convergence</u>

<u>Ex</u>: 1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has $R = \infty$.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n (-1)^{n+1}}{n}$ has $R = 1$.

#10:

$$1-\frac{2!}{1\cdot 3}+\frac{3!}{1\cdot 3\cdot 5}-\frac{4!}{1\cdot 3\cdot 5\cdot 7}+\frac{5!}{1\cdot 3\cdot 5\cdot 7\cdot 9}\cdots$$

$$\cdots+(-1)^{n-1}\frac{n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(-1)^n(n+1)!/(1\cdot 3\cdot 5\cdots\cdot(2(n+1)-1))}{(-1)^{n-1}n!/(1\cdot 3\cdot 5\cdots\cdot(2n-1))}\right|$$

$$(n+1)\cdot\cancel{n!}$$

$$=\lim_{n\to\infty}\frac{(n+1)!\cdot\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdots\cancel{(2n-1)}}{\cancel{n!}\cdot\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdots\cancel{(2n-1)}(2n+1)}$$

$$=\lim_{n\to\infty}\frac{n+1}{2n+1}=\frac{1}{2}<1$$

$$\Rightarrow\ \boxed{\text{series conv. abs.}}$$

#11:

$$\sum_{n=1}^{\infty}\frac{(-1)^n 5^n n!}{9\cdot 16\cdot\cdots\cdot(7n+2)}$$

Ratio Test:

$$\left| \frac{\cancel{(-1)^{n+1}}\, 5^{\cancel{n+1}}\, \cancel{(n+1)!}^{\,n+1} \Big/ \big(9\cdot 16 \cdots (7(n+1)+2)\big)}{\cancel{(-1)^{n}}\, \cancel{5^{n}}\, \cancel{n!} \Big/ \big(9\cdot 16 \cdots (7n+2)\big)} \right|$$

$$= \frac{5(n+1)\,\big(\cancel{9\cdot 16 \cdots (7n+2)}\big)}{\cancel{9\cdot 16 \cdots (7n+2)}\,(7(n+1)+2)}$$

$$= \frac{5(n+1)}{7(n+1)+2} \longrightarrow \frac{5}{7} \checkmark$$

Power Series $\longrightarrow$ P.S.

↳ These have the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^{\underline{\underline{n}}}$$

coeffs.

variable

center

Ex: The geom. series

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1 \cdot (x-0)^n$$

is a P.S. w/ center $a = \underline{\quad 0 \quad}$

and coeffs. $c_n = \underline{\quad 1 \quad}$.

Fund. Questions:

Today 1. What is the domain (of convergence) of a P.S.?

Next Time 2. What Calculus properties (e.g. cont., diff., etc.) do P.S. have?

Ex:

1. For the geom. series we know

$$\sum_{n=0}^{\infty} x^n \quad \text{conv. iff} \quad |x| < 1.$$

2. For what x does $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

$n=0$: $x^0 = 1$ even for $\underline{\underline{x=0}}$.

$0! = 1$

$\left(\text{This is a P.S. w/ center } a = \underline{\ 0\ } \right.$

$\left. \text{and coeffs. } c_n = \underline{\ 1/n!\ }. \right)$

<u>Ratio Test:</u>

$$\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} \frac{|x|^{n+1} \cdot n!}{|x|^n (n+1)!}$$

$(n+1)! = (n+1)\cdot n!$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

For <u>every</u> real x.

$\Rightarrow$ $\boxed{\text{Series converges (abs.) for all real } x.}$

3. Same for $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n(-1)^{n+1}}{n}$.

(This is a P.S. w/ center $a =$ 1
and coeffs. $c_n = (-1)^{n+1}/n$.)

Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-1)^n(-1)^{n+1}}{n}\right|}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{|x-1|^n \cdot 1}{n}}$$

$$= \lim_{n\to\infty} \frac{|x-1|}{\sqrt[n]{n}} = \frac{|x-1|}{1} = |x-1|$$

Concl.:

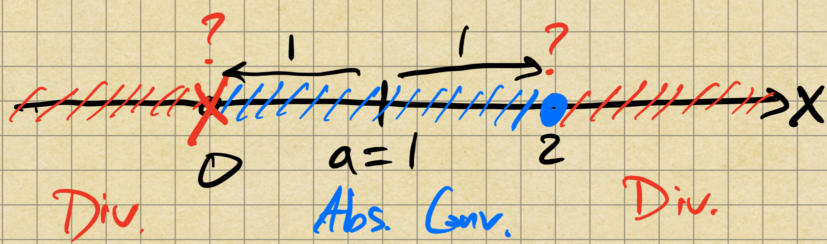
dist. between x, 1

1. $|x-1| < 1 \Rightarrow$ abs. conv.

2. $|x-1| > 1 \Rightarrow$ div.

3. $|x-1| = 1 \Rightarrow$ no info.

Picture:



x = 0: Series becomes

$$\sum_{n=1}^{\infty} \frac{(0-1)^n(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$$

↳ − (harmonic series)

Div.

x = 2: Series becomes

$$\sum_{n=1}^{\infty} \frac{(2-1)^n(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

↳ AST w/ $1/n \searrow 0$

⇒ Conv.

Thus: The domain (of convergence) is

$\boxed{x \in [0, 2].}$

In general:

Theorem: Given a P.S. $\sum_{n=0}^{\infty} c_n (x-a)^n$, there is a value $0 \le R \le \infty$ (the radius of convergence) so that:

1. $|x-a| < R \Rightarrow$ series conv. abs.

2. $|x-a| > R \Rightarrow$ series div.

[Note: Conv. @ $|x-a| = R$ must be determined separately].

Ex: 1. $\sum_{n=0}^{\infty} x^n$ conv. for $|x| < 1$, div. o.w.

$|x-0| < 1 \Rightarrow \boxed{R = 1}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ conv. for <u>all</u> x

$$\Rightarrow \boxed{R = \infty}$$

Center

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n(-1)^{n+1}}{n}$ conv. for $|x-1|<1$,

div. $|x-1|>1$ R

$$\Rightarrow \boxed{R = 1}$$

<u>Generic Picture</u>: